ZHANG WENPENG

editor

SCIENTIA MAGNA

International Book Series

Vol. 7, No. 4

2011

Editor:

Dr. Zhang Wenpeng
Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R.China

Scientia Magna

– international book series (vol. 7, no. 4) –

The Educational Publisher

2011

This book can be ordered from:

The Educational Publisher Inc.
1313 Chesapeake Ave.
Columbus, Ohio 43212
USA
Toll Free: 1-866-880-5373
E-mail: info@edupublisher.com
Website: www.EduPublisher.com

Copyright 2011 by editors and authors

Many books can be downloaded from the following
Digital Library of Science:
http://www.gallup.unm.edu/~smarandache/eBook-otherformats.htm

ISBN: 9781599731810

Printed in USA and China

Information for Authors

Papers in electronic form are accepted. They can be e-mailed in Microsoft Word XP (or lower), WordPerfect 7.0 (or lower), LaTeX and PDF 6.0 or lower.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. They must be original work and camera ready [typewritten/computerized, format: 8.5 × 11 inches (21.6 × 28 cm)]. They are not returned, hence we advise the authors to keep a copy.

The title of the paper should be writing with capital letters. The author's name has to apply in the middle of the line, near the title. References should be mentioned in the text by a number in square brackets and should be listed alphabetically. Current address followed by e-mail address should apply at the end of the paper, after the references.

The paper should have at the beginning an abstract, followed by the keywords.

All manuscripts are subject to anonymous review by three independent reviewers.

Every letter will be answered.

Each author will receive a free copy of this international book series.

Contributing to Scientia Magna book series

Authors of papers in science (mathematics, physics, engineering, philosophy, psychology, sociology, linguistics) should submit manuscripts to the main editor:

Prof. Dr. Zhang Wenpeng, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China, E-mail: wpzhang@nwu.edu.cn .

Associate Editors

Dr. W. B. Vasantha Kandasamy, Department of Mathematics, Indian Institute of Technology, IIT Madras, Chennai - 600 036, Tamil Nadu, India.

Dr. Larissa Borissova and Dmitri Rabounski, Sirenevi boulevard 69-1-65, Moscow 105484, Russia.

Dr. Zhaoxia Wu, School of Applied Mathematics, Xinjiang University of Finance and Economics, Urmq, P.R.China. E-mail: wuzhaoxia828@163.com

Prof. Yuan Yi, Research Center for Basic Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P.R.China. E-mail: yuanyi@mail.xjtu.edu.cn .

Dr. Zhefeng Xu, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China. E-mail: zfxu@nwu.edu.cn .

Prof. József Sándor, Babes-Bolyai University of Cluj, Romania. E-mail: jjsandor@ hotmail.com; jsandor@member.ams.org

Dr. Xia Yuan, Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China. E-mail: yuanxia11@163.com

Contents

R. Ma and Y. Zhang : On Erdös-Smarandache numbers 1

R. Dhavaseelan, etc. : A view on generalized intuitionistic fuzzy contra continuous functions and its applications 7

L. Huan : On the asymptotic properties of triangular base sequence 19

A. Esi : A new class of double interval numbers 23

M. Ren, etc. : Linear operators preserving commuting pairs of matrices over semirings 29

S. S. Billing : Conditions for a subclass of analytic functions 35

B. Hazarika : On ideal convergence in topological groups 42

Salahuddin : A summation formula including recurrence relation 49

A. Esi and E. Eren : Some classes of difference fuzzy numbers defined by an orlicz function 82

M. Dragan and M. Bencze : The Hadwiger-Finsler reverse in an acute triangle 92

Q. Yang and C. Fu : On mean value computation of a number theoretic function 96

Y. Lu, etc. : On Kummer's fourier series for $\log\Gamma(x)$ 104

J. Sándor : On two inequalities for the composition of arithmetic functions 109

G. Mirhosseinkhani : Compactness and proper maps in the category of generated spaces 114

Scientia Magna
Vol. 7 (2011), No. 4, 1-6

On Erdös-Smarandache numbers[1]

Rong Ma† and Yulong Zhang‡

† School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, People's Republic of China
‡ School of Electronic and Information Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, People's Republic of China
E-mail: marong@nwpu.edu.cn zzboyzyl@163.com

Abstract Let A denote the set of Erdös-Smarandache numbers, for any positive integer n, they can be defined by the solutions of the diophantine equation $P(n) = S(n)$, where $P(n)$ is the largest prime factor which divides n, $S(n)$ is the Smarandache function defined as follows: $S(n)$ is the smallest number, such that $S(n)!$ is divisible by n, i.e., $S(n) = \min\{m : m \in N,\ n|m!\}$. For any real number $x > 1$, let $M(x)$ denote the number of natural numbers n in the set A and such that $n \leq x$, in this paper, we study $M(x)$ by the elementary and analytic methods, and give an interesting asymptotic formula of $M(x)$.
Keywords Erdös-Smarandache numbers, the largest prime factor, asymptotic formula.
MR(2000) Classification: 11N25, 11N37.

§1. Introduction

For any positive integer n, the Smarandache function $S(n)$ can be defined as follows: $S(n)$ is the smallest number, such that $S(n)!$ is divisible by n, i.e., $S(n) = \min\{m : m \in N,\ n|m!\}$. According to the definition of $S(n)$, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ denoting the prime powers factorization of n, we have $S(n) = \max_{1\leq i\leq r}\{S(p_i^{\alpha_i})\}$. Hence we can easily get several values of $S(n)$: $S(1) = 1,\ S(2) = 2,\ S(3) = 3,\ S(4) = 4,\ S(5) = 5,\ S(6) = 3,\ S(7) = 7,\ S(8) = 4,\ S(9) = 6,\ \cdots\cdots$. About the arithmetical properties of $S(n)$, many scholars have studied them (see Ref.[4]-[8]). For example, Lu (see Ref.[4]) and Le (see Ref.[5]) studied the solutions involving the equations of $S(n)$, i.e., the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite many positive integer solutions.

Du (see Ref.[6]) studied the conjecture on $S(n)$, i.e., when n was a sqarefree number, the sum

$$\sum_{d|n} \frac{1}{S(n)}$$

[1]This work is supported by the 2011 Basic Research Fund of Northwestern Polytechnical University(JC201123).

is impossible an integer.

Xu (see Ref.[7]) got a profound result about $S(n)$, that is

$$\sum_{n\leq x}(S(n)-P(n))^2=\frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $P(n)$ is the largest prime factor which divides n, $\zeta(s)$ is Riemann zeta-function.

Now we let A denote the set of Erdös-Smarandache numbers (See Ref.[8] and [9]), which is defined by the solutions of the diophantine equation $S(n)=P(n)$. According to the definitions of $S(n)$ and $P(n)$, we can easily get several front Erdös-Smarandache numbers, i.e., $A=\{2,3,5,6,7,10,11,13,14,15,17,19,21,22,23,26,28,29,30,31,\cdots\}$. In 1991, for any positive integer $k>3$, Erdös (see Ref. [10]) asserted the number of elements of the set A was

$$\sum_{\substack{n\leq k\\ n\in A}}1=k+o(k)\quad(k\to+\infty).$$

After that, many scholars (see Ref.[11,12]) improved the result. Further more, in 2005, Ivic (see Ref.[1]) got the asymptotic formula

$$\sum_{\substack{n\leq k\\ n\notin A}}1=k\exp\left\{-\sqrt{2\log k\log_2 k}\left(1+O\left(\frac{\log_3 k}{\log_2 k}\right)\right)\right\},$$

where $\exp(x)=e^x$. Even he got the asymptotic formula involving the Dickman-de Bruijn function. Thereby he sharpened and corrected results of Ford (see Ref.[2]), Koninck and Doyon (see Ref.[3]).

The authors are also very interested in this problem and want to know some more about the number of Erdös-Smarandache numbers. So we use the elementary method to study this problem and get a weaker asymptotic formula of the number of Erdös-Smarandache numbers. Although the result is weaker than those of the above, we have used the alternative approach (completely different from the Ivic's). That is, we shall prove the following:

Theorem 1.1. Let A denote the set of Erdös-Smarandache numbers, then for all positive integer $k>2$, we have the asymptotic formula

$$\sum_{\substack{n\leq k\\ n\in A}}1=k+O\left(\frac{k}{\log k(\log\log k)^{\frac{3}{2}}}\right).$$

§2. Some lemmas

To complete the proof of the above theorem, we need the several following lemmas. First, we give the familiar formula Abel's identity.

Lemma 2.1. Abel's identity. For any arithmetical function $a(n)$, let

$$A(x)=\sum_{n\leq x}a(n),$$

where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[x,\ y]$, where $0 < x < y$. Then we have

$$\sum_{x<n\leq y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)dt. \tag{1}$$

Proof. See Ref.[13].

Lemma 2.2. Prime number theorem. For $x > 0$, let $\pi(x)$ denote the number of primes not exceeding x. Then we have

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{2}$$

In addition, the prime number theorem also has an equivalent form

$$\vartheta(x) = \sum_{p\leq x} \log p = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt = x + O\left(\frac{x}{\log x}\right). \tag{3}$$

Proof. See Ref.[14].

Lemma 2.3. For all $0 < x < y$, we have

$$\sum_{x<p\leq y} \frac{1}{p^2} = \frac{1}{x\log x} + O\left(\frac{1}{x\log^2 x}\right).$$

Proof. Let $a(n)$ denote the characteristic function of the primes, that is

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{x<p\leq y} \frac{1}{p^2} = \sum_{x<n\leq y} \frac{a(n)}{n^2}.$$

Therefore if we take $f(t) = \frac{1}{t^2}$ in Lemma 2.1, we find,

$$\sum_{x<p\leq y} \frac{1}{p^2} = \frac{1}{y^2}\sum_{n\leq y} a(n) - \frac{1}{x^2}\sum_{n\leq x} a(n) + 2\int_x^y \frac{1}{t^3}\sum_{n\leq t} a(n)dt. \tag{4}$$

From Lemma 2.2 we have $\pi(x) = \sum\limits_{n\leq x} a(n) = \frac{x}{\log x} + R(x)$, where $R(x) = O(\frac{x}{\log^2 x})$, then we find

$$\begin{aligned}
\sum_{x<p\leq y} \frac{1}{p^2} &= \frac{\frac{y}{\log y} + R(y)}{y^2} - \frac{\frac{x}{\log x} + R(x)}{x^2} + 2\int_x^y \frac{\frac{t}{\log t} + R(t)}{t^3}dt \\
&= \frac{1}{y\log y} + \frac{R(y)}{y^2} - \frac{1}{x\log x} - \frac{R(x)}{x^2} + 2\int_x^y \frac{1}{t^2\log t}dt + 2\int_x^y \frac{R(t)}{t^3}dt \\
&= \frac{1}{y\log y} - \frac{1}{x\log x} + O\left(\frac{1}{x\log^2 x}\right) + 2\int_x^y \frac{1}{t^2\log t}dt + 2\int_x^y \frac{R(t)}{t^3}dt \\
&= -\frac{1}{x\log x} + O\left(\frac{1}{x\log^2 x}\right) + 2\int_x^y \frac{1}{t^2\log t}dt + 2\int_x^y \frac{R(t)}{t^3}dt.
\end{aligned} \tag{5}$$

Now

$$\begin{aligned}\int_x^y \frac{1}{t^2\log t}dt &= \frac{1}{x\log x}-\frac{1}{y\log y}-\int_x^y \frac{1}{t^2\log^2 t}dt \\ &= \frac{1}{x\log x}+O\left(\frac{1}{x\log^2 x}\right) \end{aligned} \tag{6}$$

and

$$\int_x^y \frac{R(t)}{t^3}dt = O\left(\int_x^y \frac{1}{t^2\log^2 t}dt\right) = O\left(\frac{1}{x\log^2 x}\right), \tag{7}$$

where we used the condition $R(t)=O(\frac{t}{\log^2 t})$. Hence equation (5) can be expressed as follows:

$$\sum_{x<p\le y}\frac{1}{p^2} = \frac{1}{x\log x}+O\left(\frac{1}{x\log^2 x}\right).$$

This proves the lemma.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we will estimate the upper bound of $\sum_{\substack{n\le k\\ n\notin A}} 1$. In fact, for any positive integer $k>1$, we let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ denote the prime powers factorization of n, then according to the definitions and properties of $S(n)$ and $P(n)$, we can let $S(n)=S(p_i^{\alpha_i})=mp_i$, where $S(p_i^{\alpha_i})$ is the largest number of $\{S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \cdots,\ S(p_r^{\alpha_r})\}$. If $\alpha_i=1$, then $m=1$ and $S(n)=p_i$ is prime. If $\alpha_i>1$, then $m>1$ and $S(n)$ is composite. Hence $\sum_{\substack{n\le k\\ n\notin A}} 1$ is the number of n $(1\le n\le k)$ such that $S(n)=1$ or $S(n)$ is composite. Apparently $S(n)=1$ if and only if $n=1$. Therefore we have

$$\sum_{\substack{n\le k\\ n\notin A}} 1 = 1+\sum_{\substack{n\le k\\ S(n)=S(p^\alpha),\,\alpha\ge 2}} 1 \le 1+\sum_{\substack{np^\alpha\le k\\ \alpha p>mp>M,\,\alpha\ge 2}} 1+\sum_{S(n)\le M} 1, \tag{8}$$

where $M\ll k^\epsilon$, $\epsilon>0$ is a real number. Now we will estimate every term of (8). Firstly we will estimate the first summation. From Lemma 2.3 we have

$$\begin{aligned}\sum_{\substack{np^\alpha\le k\\ \alpha p>M,\,\alpha\ge 2}} 1 &= \sum_{\substack{np^2\le k\\ 2p>M}} 1+\sum_{\substack{np^\alpha\le k\\ \alpha p>M,\,\alpha\ge 3}} 1 = \sum_{\frac{M}{2}<p\le\sqrt{k}}\ \sum_{n\le\frac{k}{p^2}} 1+\sum_{\substack{p^\alpha\le k\\ \alpha p>M,\,\alpha\ge 3}}\ \sum_{n\le\frac{k}{p^\alpha}} 1 \\ &= \sum_{\frac{M}{2}<p\le\sqrt{k}}\left(\frac{k}{p^2}+O(1)\right)+\sum_{\substack{p^\alpha\le k\\ \alpha p\ge M,\,3\le\alpha<p}}\left(\frac{k}{p^\alpha}+O(1)\right)+\sum_{\substack{p^\alpha\le k\\ \alpha p>M,\,\alpha\ge p}}\left(\frac{k}{p^\alpha}+O(1)\right) \\ &\le \frac{2k}{M\log M}+O\left(\frac{k}{M\log^2 M}\right)+\sum_{\substack{\sqrt{M}<p\le\sqrt{k}\\ \alpha\ge 3}}\left(\frac{k}{p^\alpha}+O(1)\right)+\sum_{\substack{p\le\sqrt{k}\\ \alpha>\sqrt{M}}}\left(\frac{k}{p^\alpha}+O(1)\right) \\ &\ll \frac{2k}{M\log M}+O\left(\frac{k}{M\log^2 M}\right)+\frac{k}{M\sqrt{M}}+\frac{k}{2^{\sqrt{M}-1}} \ll \frac{k}{M\log M}, \end{aligned} \tag{9}$$

where $M\ll k^\epsilon$.

In order to estimate the other term of (8), we must estimate it in a different way. For any prime $p\le M$, let $\alpha(p)=[\frac{M}{p-1}]$ denote the largest number not exceeding $[\frac{M}{p-1}]$. Let

$u = \prod_{p\leq M} p^{\alpha(p)}$. For any positive integer n such that $S(n) \leq M$, let $S(n) = S(p^{\alpha})$, according to the definition of $S(n)$, we must have $p^{\alpha}|M!$, thus we have $\alpha \leq \sum_{j=1}^{\infty}[\frac{M}{p^j}] \leq [\frac{M}{p-1}]$. Hence for all positive integer n such that $S(n) \leq M$, we have $n|u$, that is, the number of all the n like this is not exceeding all the divisors of u, then we have

$$\begin{aligned}\sum_{S(n)\leq M} 1 &\leq \sum_{d|u} 1 = \prod_{p\leq M}(1+\alpha(p)) = \prod_{p\leq M}\left(1+\left[\frac{M}{p-1}\right]\right)\\ &= \exp\left(\sum_{p\leq M}\log\left(1+\left[\frac{M}{p-1}\right]\right)\right), \qquad (10)\end{aligned}$$

where $\exp(y) = e^y$.

From Lemma 2.2 and (10), we have

$$\begin{aligned}\sum_{S(n)\leq M} 1 &\leq \exp\left(\sum_{p\leq M}\log\left(1+\frac{M}{p-1}\right)\right)\\ &= \exp\left\{\sum_{p\leq M}\left[\log(p-1+M)-\log p-\log\left(1-\frac{1}{p}\right)\right]\right\}\\ &\leq \exp\left(\pi(M)\log(2M)-\sum_{p\leq M}\log p+\sum_{p\leq M}\frac{1}{p}\right)\\ &= \exp\left(\frac{M\log(2M)}{\log M}-M+O\left(\frac{M}{\log M}\right)\right)\\ &\ll \exp\left(\frac{M}{\log M}\right), \qquad (11)\end{aligned}$$

where $M \ll k^{\epsilon}$. Now let $M = \log k\sqrt{\log\log k}$ (it determines $k > 2$), we have $\exp\left(\frac{M}{\log M}\right) \ll \frac{k}{\log k(\log\log k)^{\frac{3}{2}}}$, thus combining (8), (9) and (11), we immediately get

$$\sum_{\substack{n\leq k\\ n\notin A}} 1 = O\left(\frac{k}{\log k(\log\log k)^{\frac{3}{2}}}\right).$$

Therefore, we have

$$\sum_{\substack{n\leq k\\ n\in A}} 1 = k - \sum_{\substack{n\leq k\\ n\notin A}} 1 = k + O\left(\frac{k}{\log k(\log\log k)^{\frac{3}{2}}}\right).$$

This completes the proof of the theorem.

References

[1] A. Ivic, On a Problem of Erdös Involving the largest Prime Factor of n, Monatsh. Math., **145**(2005), 35-46.

[2] K. Ford, The normal behavior of the Smarandache function, Smarandache Notions Journal, **10**(1999), 81-86.

[3] J-M De Koninck and N. Doyon, On a thin set of integers involving the largest prime factor function, Int. J. Math. Sci., **19**(2003), 1185-1192.

[4] Y. Lu, On the solutions of an equation involving the Smarandache function, Scientia Magna, **2**(2006), No. 1, 76-79.

[5] M. Le, A conjecture concerning the Smarandache function, Journal of natural science of Heilongjiang University, **24**(2007), No. 5, 687-688.

[6] F. Du, On a conjecture of the Smarandache function $S(n)$, Pure and Applied Mathematics, **23**(2007), No. 2, 205-208.

[7] Z. Xu, On the value distribution of the Smarandache function, Acta Mathematica Sinica, **49**(2006), No. 5, 1009-1012.

[8] M. Le, Two function equations, Smarandache Notions Journal, **14**(2004), 180-182.

[9] P. Erdös and P. C. Ashbache, Thoughts of Pal Erdös on Some Smarandache Notions, Smarandache Notion Journal, **8**(1997), 220-224.

[10] P. Erdös, Problem 6674, Amer Math Monthly, **98**(1991), 965.

[11] I. Kastanas, Solution to Problem 6674, Amer Math Monthly, **98**(1994), 179.

[12] S. Akbik, On a density problem of Erdös, Int J Math Sci., **22**(1999), 655-658.

[13] C. Pan and C. Pan, Element of the Analytic Number Theory, Science Press, Beijing, 1991 (in Chinese).

[14] T. Apstol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[15] F. Smarandache, Definitions, Solved and Unsolved Problems, Conjectures and Theorems in Number Theory and Geometry, Chicago, Xiquan Publishing House, 2000.

[16] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

Scientia Magna
Vol. 7 (2011), No. 4, 7-18

A view on generalized intuitionistic fuzzy contra continuous functions and its applications

R. Dhavaseelan†, E. Roja‡ and M. K. Uma♯

† Department of Mathematics, Sona College of Technology, Salem, 636005, Tamil Nadu, India
‡ ♯ Department of Mathematics, Sri Saradha College for Women, Salem-16, Tamil Nadu, India
E-mail: dhavaseelan.r@gmail.com

Abstract In this paper the following concepts are generalized intuitionistic fuzzy contra continuous function, strongly generalized intuitionistic fuzzy contra continuous function and generalized intuitionistic fuzzy contra irresolute are studied. The concepts of generalized intuitionistic fuzzy S-closed and strongly generalized intuitionistic fuzzy S-closed are studied. The concepts of generalized intuitionistic fuzzy compact spaces and generalized intuitionistic fuzzy almost compact spaces are established. The concepts of generalized intuitionistic fuzzy filter and intuitionistic fuzzy C-convergent are established. Some properties are investigated with some illustrations.

Keywords Generalized intuitionistic fuzzy contra continuity, strongly generalized intuitioni -stic fuzzy contra continuity, generalized intuitionistic fuzzy contra irresolute, generalized int -uitionistic fuzzy S-closed, generalized intuitionistic fuzzy compact spaces, generalized intuiti -onistic fuzzy almost compact spaces, generalized intuitionistic fuzzy filter and intuitionistic fuzzy C-convergent.
2000 Mathematics Subject Classification: 54A40, 03E72.

§1. Introduction

The fuzzy concept has invaded almost all branches of mathematics ever since the introduction of fuzzy sets by L. A. Zadeh [16]. These fuzzy sets have applications in many fields such as information [14] and control [15]. The theory of fuzzy topological space was introduced and developed by C. L. Chang [7] and since then various notions in classical topology has extended the fuzzy topological space.

The idea of "intuitionistic fuzzy set" was first published by Atanassov [1] and many works by the same author and his colleagues appeared in the literature [2,3,4]. Later, this concept was generalized to "intuitionistic L - fuzzy sets" by Atanassov and Stoeva [5]. The concepts of "On some generalizations of fuzzy continuous functions" was introduced by G. Balasubramanian and P. Sundaram [6]. The concepts of "Generalized intuitionistic fuzzy closed sets" was introduced by R. Dhavaseelan, E. Roja and M. K. Uma [9]. The concepts of "fuzzy contra continuous" was

introduced by E. Ekici and E. Kerre [10].

In this paper the following concepts are generalized intuitionistic fuzzy contra continuous function, strongly generalized intuitionistic fuzzy contra continuous function and generalized intuitionistic fuzzy contra irresolute are studied. The concepts of generalized intuitionistic fuzzy S-closed and strongly generalized intuitionistic fuzzy S-closed are studied. The concepts of generalized intuitionistic fuzzy compact spaces and generalized intuitionistic fuzzy almost compact spaces are established. The concepts of generalized intuitionistic fuzzy filter and intuitionistic fuzzy C-convergent are established. Some properties are investigated with some illustrations.

§2. Preliminaries

Definition 2.1.[3] Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS for short) A is an object having the form $A = \{\langle x,\ \mu_A(x),\ \delta_A(x)\rangle :\ x \in X\}$ where the function $\mu_A :\ X \to I$ and $\delta_A :\ X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership ($\delta_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \delta_A(x) \leq 1$ for each $x \in X$.

Definition 2.2.[3] Let X be a nonempty set and the intuitionistic fuzzy sets A and B in the form $A = \{\langle x,\ \mu_A(x),\ \delta_A(x)\rangle :\ x \in X\}$, $B = \{\langle x,\ \mu_B(x),\ \delta_B(x)\rangle :\ x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\delta_A(x) \geq \delta_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $\bar{A} = \{\langle x,\ \delta_A(x),\ \mu_A(x)\rangle :\ x \in X\}$;

(d) $A \cap B = \{\langle x,\ \mu_A(x) \wedge \mu_B(x),\ \delta_A(x) \vee \delta_B(x)\rangle :\ x \in X\}$;

(e) $A \cup B = \{\langle x,\ \mu_A(x) \vee \mu_B(x),\ \delta_A(x) \wedge \delta_B(x)\rangle :\ x \in X\}$;

(f) $[\ ]A = \{\langle x,\ \mu_A(x),\ 1 - \mu_A(x)\rangle :\ x \in X\}$;

(g) $\langle\ \rangle A = \{\langle x,\ 1 - \delta_A(x),\ \delta_A(x)\rangle :\ x \in X\}$.

Definition 2.3.[3] Let $\{A_i :\ i \in J\}$ be an arbitrary family of intuitionistic fuzzy sets in X. Then

(a) $\bigcap A_i = \{\langle x,\ \wedge\mu_{A_i}(x),\ \vee\delta_{A_i}(x)\rangle :\ x \in X\}$;

(b) $\bigcup A_i = \{\langle x,\ \vee\mu_{A_i}(x),\ \wedge\delta_{A_i}(x)\rangle :\ x \in X\}$.

Since our main purpose is to construct the tools for developing intuitionistic fuzzy topological spaces, we must introduce the intuitionistic fuzzy sets $0_\sim$ and $1_\sim$ in X as follows:

Definition 2.4.[3] $0_\sim = \{\langle x,\ 0,\ 1\rangle :\ x \in X\}$ and $1_\sim = \{\langle x,\ 1,\ 0\rangle :\ x \in X\}$.

Here are the basic properties of inclusion and complementation:

Corollary 2.1.[3] Let $A,\ B,\ C$ be intuitionistic fuzzy sets in X. Then

(a) $A \subseteq B$ and $C \subseteq D \Rightarrow A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$;

(b) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq B \cap C$;

(c) $A \subseteq C$ and $B \subseteq C \Rightarrow A \cup B \subseteq C$;

(d) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$;

(e) $\overline{A \cup B} = \overline{A} \cap \overline{B}$;

(f) $\overline{A \cap B} = \overline{A} \cup \overline{B}$;

(g) $A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}$;

(h) $\overline{\overline{(A)}} = A$;
(i) $\overline{1_\sim} = 0_\sim$;
(j) $\overline{0_\sim} = 1_\sim$.

Now we shall define the image and preimage of intuitionistic fuzzy sets. Let X and Y be two nonempty sets and $f : X \to Y$ be a function.

Definition 2.5.[3] (a) If $B = \{\langle y,\ \mu_B(y),\ \delta_B(y)\rangle :\ y \in Y\}$ is an intuitionistic fuzzy set in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the intuitionistic fuzzy set in X defined by $f^{-1}(B) = \{\langle x,\ f^{-1}(\mu_B)(x),\ f^{-1}(\delta_B)(x)\rangle :\ x \in X\}$.

(b) If $A = \{\langle x,\ \lambda_A(x),\ \vartheta_A(x)\rangle :\ x \in X\}$ is an intuitionistic fuzzy set in X, then the image of A under f, denoted by $f(A)$, is the intuitionistic fuzzy set in Y defined by $f(A) = \{\langle y,\ f(\lambda_A)(y),\ (1 - f(1 - \vartheta_A))(y)\rangle :\ y \in Y\}$.

Where

$$f(\lambda_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_A(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise,} \end{cases}$$

$$(1 - f(1 - \vartheta_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \vartheta_A(x), & \text{if } f^{-1}(y) \neq \emptyset; \\ 1, & \text{otherwise.} \end{cases}$$

For the sake of simplicity,let us use the symbol $f_-(\vartheta_A)$ for $1 - f(1 - \vartheta_A)$.

Corollary 2.2.[3] Let $A,\ A_i (i \in J)$ be intuitionistic fuzzy sets in $X,\ B,\ B_i (i \in K)$ be intuitionistic fuzzy sets in Y and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$;
(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
(c) $A \subseteq f^{-1}(f(A))$ { If f is injective, then $A = f^{-1}(f(A))$ };
(d) $f(f^{-1}(B)) \subseteq B$ { If f is surjective, then $f(f^{-1}(B)) = B$ };
(e) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$;
(f) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$;
(g) $f(\bigcup A_i) = \bigcup f(A_i)$;
(h) $f(\bigcap A_i) \subseteq \bigcap f(A_i)$ { If f is injective, then $f(\bigcap A_i) = \bigcap f(A_i)$};
(i) $f^{-1}(1_\sim) = 1_\sim$;
(j) $f^{-1}(0_\sim) = 0_\sim$;
(k) $f(1_\sim) = 1_\sim$, if f is surjective;
(l) $f(0_\sim) = 0_\sim$;
(m) $\overline{f(A)} \subseteq f(\overline{A})$, if f is surjective;
(n) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Definition 2.6.[13] An intuitionistic fuzzy set A of intuitionistic fuzzy topological space X is called a intuitionistic fuzzy regular closed set if $IFcl(IFint(A)) = A$.

Equivalently An intuitionistic fuzzy set A of intuitionistic fuzzy topological space X is called a intuitionistic fuzzy regular open set if $IFint(IFcl(A)) = A$.

Definition 2.7.[9] Let $(X,\ T)$ be an intuitionistic fuzzy topological space. An intuitionistic fuzzy set A in $(X,\ T)$ is said to be generalized intuitionistic fuzzy closed (in shortly GIF closed)

if $IFcl\,(A) \subseteq G$ whenever $A \subseteq G$ and G is intuitionistic fuzzy open. The complement of a GIF-closed set is GIF open.

Definition 2.8.[9] Let $(X,\ T)$ be an intuitionistic fuzzy topological space and A be an intuitionistic fuzzy set in X. Then intuitionistic fuzzy generalized closure (in short $IFGcl$) and intuitionistic fuzzy generalized interior (in short $IFGint$) of A are defined by

(a) $IFGcl(A) = \bigcap\{G :\ G$ is a GIF closed set in X and $A \subseteq G\ \}$.

(b) $IFGint(A) = \bigcup\{G :\ G$ is a GIF open set in X and $A \supseteq G\}$.

Definition 2.9.[11] A nonvoid family $\mathbb{F}$ of GIF sets on X is said to be generalized intuitionistic fuzzy filter (in short GIF filter) if

(1) $0_\sim \notin \mathbb{F}$;

(2) If $A,\ B \in \mathbb{F}$ then $A \cap B \in \mathbb{F}$;

(3) If $A \in \mathbb{F}$ and $A \subset B$ then $B \in \mathbb{F}$.

§3. Generalized intuitionistic fuzzy contra continuous function

Definition 3.1. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces.

(a) A map $f :\ (X,\ T) \to (Y,\ S)$ is called intuitionistic fuzzy contra continuous (in short IF contra continuous) if the inverse image of every open set in $(Y,\ S)$ is intuitionistic fuzzy closed in $(X,\ T)$.

Equivalently if the inverse image of every closed set in $(Y,\ S)$ is intuitionistic fuzzy open in $(X,\ T)$.

(b) A map $f :\ (X,\ T) \to (Y,\ S)$ is called generalized intuitionistic fuzzy contra continuous (in short GIF contra continuous) if the inverse image of every open set in $(Y,\ S)$ is GIF closed in $(X,\ T)$.

Equivalently if the inverse image of every closed set in $(Y,\ S)$ is GIF open in $(X,\ T)$.

(c) A map $f :\ (X,\ T) \to (Y,\ S)$ is called GIF contra irresolute if the inverse image of every GIF closed set in $(Y,\ S)$ is GIF open in $(X,\ T)$.

Equivalently if the inverse image of every GIF open set in $(Y,\ S)$ is GIF closed in $(X,\ T)$.

(d) A map $f :\ (X,\ T) \to (Y,\ S)$ is said to be strongly GIF contra continuous if the inverse image of every GIF open set in $(Y,\ S)$ is intuitionistic fuzzy closed in $(X,\ T)$.

Equivalently if the inverse image of every GIF closed set in $(Y,\ S)$ is intuitionistic fuzzy open in $(X,\ T)$.

Proposition 3.1. Let $f :\ (X,\ T) \to (Y,\ S)$ be a bijective map. Then f is GIF contra continuous mapping if $IFcl(f(A)) \subseteq f(IFGint(A))$ for every intuitionistic fuzzy set A in $(X,\ T)$.

Proposition 3.2. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. Let $f :\ (X,\ T) \to (Y,\ S)$ be a map. Suppose that one of the following properties hold.

(a) $f(IFGcl(A)) \subseteq IFint(f(A))$ for each intuitionistic fuzzy set A in $(X,\ T)$.

(b) $IFGcl(f^{-1}(B)) \subseteq f^{-1}(IFint(B))$ for each intuitionistic fuzzy set B in $(Y,\ S)$.

(c) $f^{-1}(IFcl(B)) \subseteq IFGint(f^{-1}(B))$ for each intuitionistic fuzzy set B in $(Y,\ S)$.

Then f is GIF contra continuous mapping.

Proposition 3.3. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ be a map. Suppose that one of the following properties hold.

(a) $f^{-1}(IFGcl(A)) \subseteq IFGint(IFGcl(f^{-1}(A))$ for each intuitionistic fuzzy set A in $(Y,\ S)$.

(b) $IFGcl(IFGint(f^{-1}(A))) \subseteq f^{-1}(IFGint(A))$ for each intuitionistic fuzzy set A in $(Y,\ S)$.

(c) $f(IFGcl(IFGint(A))) \subseteq IFGint(f(A))$ for each intuitionistic fuzzy set A in $(X,\ T)$.

(d) $f(IFGcl(A)) \subseteq IFGint(f(A))$ for each intuitionistic fuzzy set A in $(X,\ T)$.

Then f is GIF contra continuous mapping.

Proposition 3.4. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. If $f:\ (X,\ T) \to (Y,\ S)$ is intuitionistic fuzzy contra continuous then it is GIF contra continuous.

The converse of Proposition 3.4 is not true. See Example 3.1.

Example 3.1. Let $X = \{a,\ b,\ c\}$. Define intuitionistic fuzzy sets A and B as follows

$$A = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.4}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.4})\rangle$$

and

$$B = \langle x,\ (\tfrac{a}{0.3},\ \tfrac{b}{0.4},\ \tfrac{c}{0.3}),\ (\tfrac{a}{0.5},\ \tfrac{b}{0.6},\ \tfrac{c}{0.7})\rangle.$$

Then $T = \{0_\sim,\ 1_\sim,\ A\}$ and $S = \{0_\sim,\ 1_\sim,\ B\}$ are intuitionistic fuzzy topologies on X. Thus $(X,\ T)$ and $(X,\ S)$ are intuitionistic fuzzy topological spaces. Define $f:\ (X,\ T) \to (X,\ S)$ as $f(a) = b,\ f(b) = a,\ f(c) = c$. Clearly f is GIF contra continuous. But f is not intuitionistic fuzzy contra continuous. Since, $f^{-1}(B) \notin T$ for $B \in S$.

Proposition 3.5. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. If $f:\ (X,\ T) \to (Y,\ S)$ is GIF contra irresolute then it is GIF contra continuous.

The converse of Proposition 3.5 is not true. See Example 3.2.

Example 3.2. Let $X = \{a,\ b,\ c\}$. Define the intuitionistic fuzzy sets $A,\ B$ and C as follows.

$$A = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.5},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5})\rangle,$$
$$B = \langle x,\ (\tfrac{a}{0.7},\ \tfrac{b}{0.6},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.3},\ \tfrac{b}{0.4},\ \tfrac{c}{0.5})\rangle$$

and

$$C = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5})\rangle.$$

$T = \{0_\sim,\ 1_\sim,\ A,\ B\}$ and $S = \{0_\sim,\ 1_\sim,\ C\}$ are intuitionistic fuzzy topologies on X. Thus $(X,\ T)$ and $(X,\ S)$ are intuitionistic fuzzy topological spaces. Define $f:\ (X,\ T) \to (X,\ S)$ as follows $f(a) = b,\ f(b) = a,\ f(c) = c$. Clearly f is GIF contra continuous. But f is not GIF contra irresolute. Since $D = \langle x,\ (\tfrac{a}{0.3},\ \tfrac{b}{0.4},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.5},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5})\rangle$ is GIF closed in $(X,\ S)$, $f^{-1}(D)$ is not GIF-open in $(X,\ T)$.

Proposition 3.6. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. If $f:\ (X,\ T) \to (Y,\ S)$ is strongly GIF contra continuous then f is intuitionistic fuzzy contra continuous.

The converse Proposition 3.6 is not true. See Example 3.3.

Example 3.3. Let $X = \{a,\ b,\ c\}$. Define the intuitionistic fuzzy sets $A,\ B$ and C as follows.

$$A = \langle x,\ (\tfrac{a}{0.3},\ \tfrac{b}{0.2},\ \tfrac{c}{0.2}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5})\rangle,$$
$$B = \langle x,\ (\tfrac{a}{0.1},\ \tfrac{b}{0.1},\ \tfrac{c}{0.1}),\ (\tfrac{a}{0.9},\ \tfrac{b}{0.9},\ \tfrac{c}{0.9})\rangle$$

and

$$C = \langle x,\ (\tfrac{a}{0.9},\ \tfrac{b}{0.9},\ \tfrac{c}{0.9}),\ (\tfrac{a}{0.1},\ \tfrac{b}{0.1},\ \tfrac{c}{0.1})\rangle.$$

$T = \{0_\sim,\ 1_\sim,\ A,\ B\}$ and $S = \{0_\sim,\ 1_\sim,\ C\}$ are intuitionistic fuzzy topologies on X. Thus $(X,\ T)$ and $(X,\ S)$ are intuitionistic fuzzy topological spaces. Define $f:\ (X,\ T) \to (X,\ S)$ as follows $f(a) = a,\ f(b) = b,\ f(c) = b$. Clearly f is intuitionistic fuzzy contra continuous. But f is not strongly GIF contra continuous. Since $D = \langle x,\ (\tfrac{a}{0.9},\ \tfrac{b}{0.99},\ \tfrac{c}{0.9}),\ (\tfrac{a}{0.1},\ \tfrac{b}{0},\ \tfrac{c}{0.1})\rangle$ is GIF open in $(X,\ S)$, $f^{-1}(D)$ is not intuitionistic fuzzy closed in $(X,\ T)$.

Proposition 3.7. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. If $f:\ (X,\ T) \to (Y,\ S)$ is strongly GIF contra continuous then f is GIF contra continuous.

The converse Proposition 3.7 is not true. See Example 3.4.

Example 3.4. Let $X = \{a,\ b,\ c\}$. Define the intuitionistic fuzzy sets $A,\ B$ and C as follows.

$$A = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.5},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5})\rangle,$$
$$B = \langle x,\ (\tfrac{a}{0.6},\ \tfrac{b}{0.7},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.3},\ \tfrac{c}{0.5})\rangle$$

and

$$C = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.5})\rangle.$$

$T = \{0_\sim,\ 1_\sim,\ A,\ B\}$ and $S = \{0_\sim,\ 1_\sim,\ C\}$ are intuitionistic fuzzy topologies on X . Thus $(X\ , T)$ and $(X,\ S)$ are intuitionistic fuzzy topological spaces. Define $f:\ (X,\ T) \to (X,\ S)$ as follows $f(a) = c,\ f(b) = c,\ f(c) = c$. Clearly f is GIF contra continuous. But f is not strongly GIF contra continuous. Since $D = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.3})\rangle$ is a GIF open set in $(X,\ S)$, $f^{-1}(D)$ is not intuitionistic fuzzy closed in $(X,\ T)$.

Proposition 3.8. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. If $f:\ (X,\ T) \to (Y,\ S)$ is strongly GIF contra continuous then f is GIF contra irresolute.

The converse Proposition 3.8 is not true. See Example 3.5.

Example 3.5. Let $X = \{a,\ b,\ c\}$. Define the intuitionistic fuzzy sets $A,\ B$ and C as follows.

$$A = \langle x,\ (\tfrac{a}{0.9},\ \tfrac{b}{0.9},\ \tfrac{c}{0.9}),\ (\tfrac{a}{0.1},\ \tfrac{b}{0.1},\ \tfrac{c}{0.1})\rangle,$$
$$B = \langle x,\ (\tfrac{a}{0.99},\ \tfrac{b}{0.99},\ \tfrac{c}{0.99}),\ (\tfrac{a}{0},\ \tfrac{b}{0.01},\ \tfrac{c}{0})\rangle$$

and

$$C = \langle x,\ (\tfrac{a}{0.9},\ \tfrac{b}{0.9},\ \tfrac{c}{0.9}),\ (\tfrac{a}{0.1},\ \tfrac{b}{0},\ \tfrac{c}{0.05})\rangle.$$

$T = \{0_\sim,\ 1_\sim,\ A,\ B\}$ and $S = \{0_\sim,\ 1_\sim,\ C\}$ are intuitionistic fuzzy topologies on X. Thus $(X,\ T)$ and $(X,\ S)$ are intuitionistic fuzzy topological spaces. Define $f:\ (X,\ T) \to (X,\ S)$ as follows $f(a) = a,\ f(b) = c,\ f(c) = b$. Clearly f is GIF contra irresolute. But f is not strongly

GIF contra continuous. Since $D = \langle x,\ (\frac{a}{0.99},\ \frac{b}{0.99},\ \frac{c}{0.99}),\ (\frac{a}{0.01},\ \frac{b}{0},\ \frac{c}{0.01})\rangle$ is a GIF open set in $(X,\ S)$, $f^{-1}(D)$ is not intuitionistic fuzzy closed in $(X,\ T)$.

Proposition 3.9. Let $(X,\ T)$, $(Y,\ S)$ and $(Z,\ R)$ be any three intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ and $g:\ (Y,\ S) \to (Z,\ R)$ be maps.

(i) If f is GIF contra irresolute and g is GIF contra continuous then $g \circ f$ is GIF continuous.

(ii) If f is GIF contra irresolute and g is GIF continuous then $g \circ f$ is GIF contra continuous.

(iii) If f is GIF irresolute and g is GIF contra continuous then $g \circ f$ is GIF contra continuous.

(iv) If f is strongly GIF contra continuous and g is GIF contra continuous then $g \circ f$ is intuitionistic fuzzy continuous.

(v) If f is strongly GIF contra continuous and g is GIF continuous then $g \circ f$ is intuitionistic fuzzy contra continuous.

(vi) If f is strongly GIF continuous and g is GIF contra continuous then $g \circ f$ is intuitionistic fuzzy contra continuous.

Definition 3.2. An intuitionistic fuzzy topological space $(X,\ T)$ is said to be intuitionistic fuzzy $T_{1/2}$ if every GIF closed set in $(X,\ T)$ is intuitionistic fuzzy closed in $(X,\ T)$.

Proposition 3.10. Let $(X,\ T)$, $(Y,\ S)$ and $(Z,\ R)$ be any three intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ and $g:\ (Y,\ S) \to (Z,\ R)$ be mapping and $(Y,\ S)$ be intuitionistic fuzzy $T_{1/2}$ if f and g are GIF contra continuous then $g \circ f$ is GIF continuous.

The Proposition 3.10 is not valid if $(Y,\ S)$ is not intuitionistic fuzzy $T_{1/2}$. See Example 3.6.

Example 3.6. Let $X = \{a,\ b,\ c\}$. Define the intuitionistic fuzzy sets $A,\ B,\ C$ and D as follows.

$$A = \langle x,\ (\tfrac{a}{0.3},\ \tfrac{b}{0.3},\ \tfrac{c}{0.4}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5})\rangle,$$
$$B = \langle x,\ (\tfrac{a}{0.6},\ \tfrac{b}{0.6},\ \tfrac{c}{0.6}),\ (\tfrac{a}{0.3},\ \tfrac{b}{0.3},\ \tfrac{c}{0.3})\rangle,$$
$$C = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.4}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.4})\rangle$$

and

$$D = \langle x,\ (\tfrac{a}{0.3},\ \tfrac{b}{0.4},\ \tfrac{c}{0.3}),\ (\tfrac{a}{0.5},\ \tfrac{b}{0.6},\ \tfrac{c}{0.7})\rangle.$$

Then the family $T = \{0_\sim,\ 1_\sim,\ A,\ B\}$, $S = \{0_\sim,\ 1_\sim,\ C\}$ and $R = \{0_\sim,\ 1_\sim,\ D\}$ are intuitionistic fuzzy topologics on X. Thus $(X,\ T)$, $(X,\ S)$ and $(X,\ R)$ are intuitionistic fuzzy topological spaces. Also define $f:\ (X,\ T) \to (X,\ S)$ as $f(a) = a,\ f(b) = b,\ f(c) = b$ and $g:\ (X,\ S) \to (X,\ R)$ as $g(a) = b,\ g(b) = a,\ g(c) = c$. Then f and g are GIF contra continuous functions. But $g \circ f$ is not GIF continuous. For D is intuitionistic fuzzy open in $(X,\ R)$. $f^{-1}(g^{-1}(D))$ is not GIF open in $(X,\ T)$. $g \circ f$ is not GIF continuous. Further $(X,\ S)$ is not intuitionistic fuzzy $T_{1/2}$.

Proposition 3.11. Let $(X,\ T)$, $(Y,\ S)$ and $(Z,\ R)$ be any three intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ and $g:\ (Y,\ S) \to (Z,\ R)$ be mapping and $(Y,\ S)$ be intuitionistic fuzzy $T_{1/2}$ if f is intuitionistic fuzzy contra continuous and g is GIF contra irresolute then $g \circ f$ is strongly GIF continuous.

The Proposition 3.11 is not valid if $(Y,\ S)$ is not intuitionistic fuzzy $T_{1/2}$. See Example 3.7.

Example 3.7. Let $X = \{a,\ b,\ c\}$. Define the intuitionistic fuzzy sets $A,\ B,\ C$ and D as follows.

$$A = \langle x,\ (\tfrac{a}{0.9},\ \tfrac{b}{0.9},\ \tfrac{c}{0.9}),\ (\tfrac{a}{0.1},\ \tfrac{b}{0.1},\ \tfrac{c}{0.1})\rangle,$$
$$B = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.4}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.5})\rangle,$$
$$C = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.5},\ \tfrac{c}{0.5}),\ (\tfrac{a}{0.4},\ \tfrac{b}{0.4},\ \tfrac{c}{0.5})\rangle$$

and

$$D = \langle x,\ (\tfrac{a}{0.4},\ \tfrac{b}{0.2},\ \tfrac{c}{0.3}),\ (\tfrac{a}{0.6},\ \tfrac{b}{0.8},\ \tfrac{c}{0.7})\rangle.$$

Then the family $T = \{0_\sim,\ 1_\sim,\ A,\ B\}$, $S = \{0_\sim,\ 1_\sim,\ C\}$ and $R = \{0_\sim,\ 1_\sim,\ D\}$ are intuitionistic fuzzy topologics on X. Thus $(X,\ T)$, $(X,\ S)$ and $(X,\ R)$ are intuitionistic fuzzy topological spaces. Also define $f:\ (X,\ T) \to (X,\ S)$ as $f(a) = a,\ f(b) = a,\ f(c) = b$ and $g:\ (X,\ S) \to (X,\ R)$ as $g(a) = c,\ g(b) = a,\ g(c) = b$. Then f is intuitionistic fuzzy contra continuous and g is GIF contra irresolute. But $g \circ f$ is not strongly GIF continuous. For D is GIF open in $(X,\ R)$. $f^{-1}(g^{-1}(D))$ is not intuitionistic fuzzy open in $(X,\ T)$. $g \circ f$ is not strongly GIF continuous. Further $(X,\ S)$ is not intuitionistic fuzzy $T_{1/2}$.

Definition 3.3. Let X be a non-empty set and $x \in X$ a fixed element in X. If $r \in I_0$, $s \in I_1$ are fixed real number such that $r+s \leq 1$ then the intuitionistic fuzzy set $x_{r,s} = \langle y,\ x_r,\ x_s\rangle$ is called an intuitionistic fuzzy point in X, where r denotes the degree of membership of $x_{r,s}$, s denotes the degree of nonmembership of $x_{r,s}$. The intuitionistic fuzzy point $x_{r,s}$ is contained in the intuitionistic fuzzy set A iff $r < \mu_A(x)$, $s > \delta_A(x)$.

Proposition 3.12. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. For a function $f:\ (X,\ T) \to (Y,\ S)$ the following statements are equivalents:

(a) f is GIF contra continuous mapping.

(b) For each intuitionistic fuzzy point $x_{r,s}$ of X and for each intuitionistic fuzzy closed set B of $(Y,\ S)$ containing $f(x_{r,s})$, there exists a GIF open set A of $(X,\ T)$ containing $x_{r,s}$, such that $A \subseteq f^{-1}(B)$.

(c) For each intuitionistic fuzzy point $x_{r,s}$ of X and for each intuitionistic fuzzy closed set B of $(Y,\ S)$ containing $f(x_{r,s})$, there exists a GIF open set A of $(X,\ T)$ containing $x_{r,s}$, such that $f(A) \subseteq B$.

Proof. (a)⇒(b) Let f is GIF contra continuous mapping. Let B be an intuitionistic fuzzy closed set in $(Y,\ S)$ and let $x_{r,s}$ be an intuitionistic fuzzy point of X, such that $f(x_{r,s}) \in B$. Then $x_{r,s} \in f^{-1}(B)$, $x_{r,s} \in f^{-1}(B) = IFGint(f^{-1}(B))$. Let $A = IFGint(f^{-1}(B))$, then A is a GIF open set and $A = IFGint(f^{-1}(B)) \subseteq f^{-1}(B)$. This implies $A \subseteq f^{-1}(B)$.

(b)⇒(c) Let B be an intuitionistic fuzzy closed set in $(Y,\ S)$ and let $x_{r,s}$ be an intuitionistic fuzzy point in X, such that $f(x_{r,s}) \in B$. Then $x_{r,s} \in f^{-1}(B)$. By hypothesis $f^{-1}(B)$ is a GIF open set in $(X,\ T)$ and $A \subseteq f^{-1}(B)$. This implies $f(A) \subseteq f(f^{-1}(B)) \subseteq B$.

(c)⇒(a) Let B be an intuitionistic fuzzy closed set in $(Y,\ S)$ and let $x_{r,s}$ be an intuitionistic fuzzy point in X, such that $f(x_{r,s}) \in B$. Then $x_{r,s} \in f^{-1}(B)$. By hypothesis there exists a GIF open set A of $(X,\ T)$, such that $x_{r,s} \in A$ and $f(A) \subseteq B$. This implies $x_{r,s} \in A \subseteq$

$f^{-1}(f(A)) \subseteq f^{-1}(B)$. Since A is GIF open, $A = IFGint(A) \subseteq IFGint(f^{-1}(B))$. Therefore $x_{r,s} \in IFGint(f^{-1}(B))$, $f^{-1}(B) = \bigcup_{x_{r,s} \in f^{-1}(B)}(x_{r,s}) \subseteq IFGint(f^{-1}(B) \subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is a GIF open set in $(X,\ T)$. Thus f is GIF contra continuous mapping.

Definition 3.4. Let $(X,\ T)$ and $(Y,\ S)$ be any intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ be a mapping. The graph $g:\ X \to X \times Y$ of f is defined by $g(x) = (x,\ f(x)),\ \forall x \in X$.

Proposition 3.13. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ be any mapping. If the graph $g:\ X \to X \times Y$ of f is GIF contra continuous then f is also GIF contra continuous.

Proof. Let A be an intuitionistic fuzzy open set in $(Y,\ S)$. By definition $f^{-1}(A) = 1_\sim \bigcap f^{-1}(A) = g^{-1}(1_\sim \times A)$. Since g is GIF contra continuous, $g^{-1}(1_\sim \times A)$ is GIF closed in $(X,\ T)$. Now $f^{-1}(A)$ is a GIF closed set in $(X,\ T)$. Thus f is GIF contra continuous.

Proposition 3.14. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ be any mapping. If the graph $g:\ X \to X \times Y$ of f is strongly GIF contra continuous then f is also strongly GIF contra continuous.

Proof. Let A be an GIF open set in $(Y,\ S)$. By definition $f^{-1}(A) = 1_\sim \bigcap f^{-1}(A) = g^{-1}(1_\sim \times A)$. Since g is strongly GIF contra continuous, $g^{-1}(1_\sim \times A)$ is intuitionistic fuzzy closed in $(X,\ T)$. Now $f^{-1}(A)$ is a intuitionistic fuzzy closed set in $(X,\ T)$. Thus f is strongly GIF contra continuous.

Proposition 3.15. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. Let $f:\ (X,\ T) \to (Y,\ S)$ be any mapping. If the graph $g:\ X \to X \times Y$ of f is GIF contra irresolute then f is also GIF contra irresolute.

Proof. Let A be an GIF open set in $(Y,\ S)$. By definition $f^{-1}(A) = 1_\sim \bigcap f^{-1}(A) = g^{-1}(1_\sim \times A)$. Since g is GIF contra irresolute, $g^{-1}(1_\sim \times A)$ is GIF closed in $(X,\ T)$. Now $f^{-1}(A)$ is a GIF closed set in $(X,\ T)$. Thus f is GIF contra irresolute.

Definition 3.5. Let $(X,\ T)$ be an intuitionistic fuzzy topological space. If a family $\{\langle x,\ \mu_{G_i},\ \delta_{G_i}\rangle :\ i \in J\}$ of GIF open sets in $(X,\ T)$ satisfies the condition $\bigcup\{\langle x,\ \mu_{G_i},\ \delta_{G_i}\rangle : i \in J\} = 1_\sim$ then it is called a GIF open cover of $(X,\ T)$.

Definition 3.6. A finite subfamily of a GIF open cover $\{\langle x,\ \mu_{G_i},\ \delta_{G_i}\rangle :\ i \in J\}$ of $(X,\ T)$ which is also a GIF open cover of $(X,\ T)$ is called a finite subcover of $\{\langle x,\ \mu_{G_i},\ \delta_{G_i}\rangle :\ i \in J\}$.

Definition 3.7. An intuitionistic fuzzy topological space $(X,\ T)$ is called GIF compact iff every GIF open cover of $(X,\ T)$ has a finite subcover.

Proposition 3.16. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces and $f:\ (X,\ T) \to (Y,\ S)$ be GIF contra continuous surjection. If $(X,\ T)$ is GIF compact then so is $(Y,\ S)$.

Proof. Let $G_i = \{\langle y,\ \mu_{G_i},\ \delta_{G_i}\rangle :\ i \in J\}$ be an intuitionistic fuzzy closed in $(Y,\ S)$. $1 - G_i$ be an intuitionistic fuzzy open cover in $(Y,\ S)$ with $\bigcup\{\langle 1-y,\ 1-\mu_{G_i},\ 1-\delta_{G_i}\rangle :\ i \in J\} = \bigcup_{i\in J} 1 - G_i = 1_\sim$. Since f is GIF contra continuous, $f^{-1}(G_i)=\{\langle x,\ \mu_{f^{-1}(G_i)},\ \delta_{f^{-1}(G_i)}\rangle : i \in J\}$ is GIF open cover of $(X,\ T)$. Now $\bigcup_{i\in J} f^{-1}(G_i) = f^{-1}(\bigcup_{i\in J} G_i) = 1_\sim$. Since $(X,\ T)$ is GIF compact, there exists a finite subcover $J_0 \subset J$ such that $\bigcup_{i\in J_0} f^{-1}(G_i) = 1_\sim$. Hence $f(\bigcup_{i\in J_0} f^{-1}(G_i)) = 1_\sim$, $ff^{-1}(\bigcup_{i\in J_0}(G_i)) = \bigcup_{i\in J_0}(1-G_i) = 1_\sim$. That is, $\bigcup_{i\in J_0}(1-G_i) = 1_\sim$. Therefore $(Y,\ S)$ is intuitionistic fuzzy compact.

Definition 3.8. (i) An intuitionistic fuzzy set A of intuitionistic fuzzy topological space X is called a GIF regular closed set if $IFGcl(IFGint(A)) = A$.

Equivalently An intuitionistic fuzzy set A of intuitionistic fuzzy topological space X is called a GIF regular open set if $IFGint(IFGcl(A)) = A$.

(ii) An intuitionistic fuzzy topological space $(X,\ T)$ is called a intuitionistic fuzzy S-closed space if each intuitionistic fuzzy regular closed cover of X has a finite subcover for X.

(iii) An intuitionistic fuzzy topological space $(X,\ T)$ is called a GIF S-closed space if each GIF regular closed cover of X has a finite subcover for X.

(iv) An intuitionistic fuzzy topological space $(X,\ T)$ is called a strongly intuitionistic fuzzy S-closed space if each intuitionistic fuzzy closed cover of X has a finite subcover for X.

(v) An intuitionistic fuzzy topological space $(X,\ T)$ is called a strongly GIF S-closed space if each GIF closed cover of X has a finite subcover for X.

Proposition 3.17. Every strongly GIF S-closed space of $(X,\ T)$ is GIF S-closed.

Proof. Let $(X,\ T)$ be strongly GIF S-closed space and let $\bigcup_{i\in J}(G_i) = 1_\sim$. Where $\{G_i\}_{i\in J}$ is a family of GIF regular closed sets in $(X,\ T)$. Since every GIF regular closed is a GIF closed set, $\bigcup_{i\in J}(G_i) = 1_\sim$ and $(X,\ T)$ is strongly GIF S-closed implies that there exists a finite subfamily $\{G_i\}_{i\in J_0\subset J}$, such that $\bigcup_{i\in J_0}(G_i) = 1_\sim$. Here the finite cover of X by GIF regular closed sets has a finite subcover. Therefore $(X,\ T)$ is GIF S-closed.

Proposition 3.18. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces and let $f:\ (X,\ T) \to (Y,\ S)$ be GIF contra continuous function. If $(X,\ T)$ is strongly GIF S-closed space then $(Y,\ S)$ is intuitionistic fuzzy compact.

Proof. Let $G_i = \{\langle y,\ \mu_{G_i}(y),\ \delta_{G_i}(y)\rangle :\ i \in J\}$ be intuitionistic fuzzy open cover of $(Y,\ S)$ and let $\bigcup_{i\in J}(G_i) = 1_\sim$. Since f is GIF contra continuous, $f^{-1}(G_i) = \{\langle x,\ \mu_{f^{-1}(G_i)}(x),\ \delta_{f^{-1}(G_i)}(x)\rangle\}$ is GIF closed cover of $(X,\ T)$ and $\bigcup_{i\in J} f^{-1}(G_i) = f^{-1}(\bigcup_{i\in J}(G_i)) = 1_\sim$. Since $(X,\ T)$ is strongly GIF S-closed, there exists a finite subcover $J_0 \subset J$, such that $\bigcup_{i\in J_0} f^{-1}(G_i) = 1_\sim$. Hence $f(\bigcup_{i\in J_0} f^{-1}(G_i)) = ff^{-1}(\bigcup_{i\in J_0}(G_i)) = \bigcup_{i\in J_0}(G_i) = 1_\sim$. Therefore $(Y,\ S)$ is intuitionistic fuzzy compact.

Proposition 3.19. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces and let $f:\ (X,\ T) \to (Y,\ S)$ be strongly GIF contra continuous function. If $(X,\ T)$ is intuitionistic fuzzy compact space then $(Y,\ S)$ is GIF S-closed.

Proof. Let $G_i = \{\langle y,\ \mu_{G_i}(y),\ \delta_{G_i}(y)\rangle :\ i \in J\}$ be GIF regular closed cover of $(Y,\ S)$. Every GIF regular closed set is GIF closed set, let $\bigcup_{i\in J}(G_i) = 1_\sim$. Since f is strongly GIF contra continuous, $f^{-1}(G_i) = \{\langle x,\ \mu_{f^{-1}(G_i)}(x),\ \delta_{f^{-1}(G_i)}(x)\rangle\}$ is intuitionistic fuzzy open cover of $(X,\ T)$ and $\bigcup_{i\in J} f^{-1}(G_i) = f^{-1}(\bigcup_{i\in J}(G_i)) = 1_\sim$. Since $(X,\ T)$ is intuitionistic fuzzy compact space, there exists a finite subcover $J_0 \subset J$, such that $\bigcup_{i\in J_0} f^{-1}(G_i) = 1_\sim$. Hence $f(\bigcup_{i\in J_0} f^{-1}(G_i)) = ff^{-1}(\bigcup_{i\in J_0}(G_i)) = \bigcup_{i\in J_0}(G_i) = 1_\sim$. Therefore $(Y,\ S)$ is GIF S-closed.

Proposition 3.20. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces and let $f:\ (X,\ T) \to (Y,\ S)$ be GIF contra irresolute. If $(X,\ T)$ is GIF compact space then $(Y,\ S)$ is strongly GIF S-closed.

Proof. Let $G_i = \{\langle y,\ \mu_{G_i}(y),\ \delta_{G_i}(y)\rangle :\ i \in J\}$ be GIF closed cover of $(Y,\ S)$ and let $\bigcup_{i\in J}(G_i) = 1_\sim$. Since f is GIF contra irresolute, $f^{-1}(G_i) = \{\langle x,\ \mu_{f^{-1}(G_i)}(x),\ \delta_{f^{-1}(G_i)}(x)\rangle\}$ is GIF open cover of $(X,\ T)$ and $\bigcup_{i\in J} f^{-1}(G_i) = f^{-1}(\bigcup_{i\in J}(G_i)) = 1_\sim$. Since $(X,\ T)$ is

GIF compact space, there exists a finite subcover $J_0 \subset J$, such that $\bigcup_{i\in J_0} f^{-1}(G_i) = 1_\sim$. Hence $f(\bigcup_{i\in J_0} f^{-1}(G_i)) = ff^{-1}(\bigcup_{i\in J_0}(G_i)) = \bigcup_{i\in J_0}(G_i) = 1_\sim$. Therefore $(Y,\ S)$ is strongly GIF S-closed.

Definition 3.9. An intuitionistic fuzzy topological space X is called almost compact space if each intuitionistic fuzzy open cover of X has finite subcover, the intuitionistic fuzzy closure of whose members cover X.

Definition 3.10. An intuitionistic fuzzy topological space X is called GIF almost compact space if each GIF open cover of X has finite subcover, the GIF closure of whose members cover X.

Proposition 3.21. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces and let $f:\ (X,\ T) \to (Y,\ S)$ be GIF contra irresolute and onto mapping. If $(X,\ T)$ is GIF compact space then $(Y,\ S)$ is GIF almost compact.

Proof. Let $G_i = \{\langle y,\ \mu_{G_i}(y),\ \delta_{G_i}(y)\rangle :\ i \in J\}$ be GIF open cover of $(Y,\ S)$. Then $1_\sim = \bigcup_{i\in J} G_i \subseteq \bigcup_{i\in J}(IFGcl(G_i))$. Since f is GIF contra irresolute, $f^{-1}(IFGcl(G_i))$ is GIF open cover of $(X,\ T)$ and $\bigcup_{i\in J} f^{-1}(IFGcl(G_i)) = 1_\sim$. Since $(X,\ T)$ is GIF compact, there exists a finite subcover $J_0 \subset J$, such that $\bigcup_{i\in J_0} f^{-1}(IFGcl(G_i)) = 1_\sim$. Hence $1_\sim = f(\bigcup_{i\in J_0} f^{-1}(IFGcl(G_i))) = ff^{-1}(\bigcup_{i\in J_0}(IFGcl(G_i))) = \bigcup_{i\in J_0}(IFGcl(G_i)$. Therefore $(Y,\ S)$ is GIF almost compact.

Definition 3.11. (i) A nonempty family $\mathbb{F}$ of GIF open sets on $(X,\ T)$ is said to be a GIF filter if

(1) $0_\sim \notin \mathbb{F}$;

(2) If $A,\ B \in \mathbb{F}$ then $A \cap B \in \mathbb{F}$;

(3) If $A \in \mathbb{F}$ and $A \subset B$ then $B \in \mathbb{F}$;

(ii) A nonempty family $\mathbb{B}$ of GIF open sets on $\mathbb{F}$ is said to be a GIF filter base if

(1) $0_\sim \notin \mathbb{B}$;

(2) If $B_1, B_2 \in \mathbb{B}$ then $B_3 \subset B_1 \cap B_2$ for some $B_3 \in \mathbb{B}$;

(iii) A GIF filter $\mathbb{F}$ is called GIF convergent to a intuitionistic fuzzy point $x_{r,s}$ of a intuitionistic fuzzy topological space $(X,\ T)$ if for each GIF open set A of $(X,\ T)$ containing $x_{r,s}$, there exists a intuitionistic fuzzy set $B \in \mathbb{F}$ such that $B \subseteq A$;

(iv) A intuitionistic fuzzy filter $\mathbb{F}$ is said to be intuitionistic fuzzy C-convergent to a intuitionistic fuzzy point $x_{r,s}$ of a intuitionistic fuzzy topological space $(X,\ T)$ if for each intuitionistic fuzzy closed set A of $(X,\ T)$ containing $x_{r,s}$, there exists a intuitionistic fuzzy set $B \in \mathbb{F}$ such that $B \subseteq A$.

Proposition 3.22. Let $(X,\ T)$ and $(Y,\ S)$ be any two intuitionistic fuzzy topological spaces. A mapping $f:\ (X,\ T) \to (Y,\ S)$ is GIF contra continuous. If for each intuitionistic fuzzy point $x_{r,s}$ and each GIF filter $\mathbb{F}$ in $(X,\ T)$ is GIF convergent to $x_{r,s}$ then intuitionistic fuzzy filter $f(\mathbb{F})$ is intuitionistic fuzzy C-convergent to $f(x_{r,s})$.

Proof. Let $x_{r,s} = \langle x,\ x_r,\ x_s\rangle$ be a intuitionistic fuzzy point and $\mathbb{F}$ be any GIF filter in $(X,\ T)$ is GIF convergent to $x_{r,s}$. Since f is GIF contra continuous mapping, for each intuitionistic fuzzy closed set A containing $f(x_{r,s})$, there exists a GIF open set B of $(X,\ T)$ containing $x_{r,s}$, such that $f(B) \subseteq A$. Since $\mathbb{F}$ is GIF convergent to $x_{r,s}$, there exists intuitionistic fuzzy set $C \in \mathbb{F}$ such that $C \subseteq B$. Then $f(C) \subseteq f(B) \subseteq A$. That is $f(C) \subseteq A$. Hence the

intuitionistic fuzzy filter $f(\mathbb{F})$ is intuitionistic fuzzy C-convergent to $f(x_{r,s})$.

References

[1] K. Atanassov, lntuitionistic fuzzy sets, V. Sgurev, Ed., VII Itkr's Session, Sofia (June 1983 Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

[2] K. Atanassov, intuitionistic fuzzy sets, Fuzzy Sets and Systems, **20**(1986), 87-96.

[3] K. Atanassov, Review and new results on intuitionistic fuzzy sets, Preprint IM-MFAIS-1-88, Sofia, 1988.

[4] K. Atanassov and S. Stoeva, intuitionistic fuzzy sets, Polish Syrup. on Interval & Fuzzy Mathematics, Poznan, August, 1983, 23-26.

[5] K. Atanassov and S. Stoeva, intuitionistic L-fuzzy sets, in: R. Trappl, Ed., Cybernetics and System Research, Elsevier, Amsterdam, **2**(1984), 539-540.

[6] G. Balasubramanian, P. Sundaram, On Some generalizations of fuzzy continuous functions, Fuzzy Sets and Systems, **86**(1997), 93-100.

[7] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., **24**(1968), 182-190.

[8] D. Coker and A. Es Haydar, On fuzzy S-closed, TU. J Math., 1987, 145-152.

[9] R. Dhavaseelan, E. Roja and M. K. Uma, Generalized intuitionistic Fuzzy Closed Sets, Advances in Fuzzy Mathematics, **5**(2010), 152-172.

[10] E. Ekici and E. Kerre, On fuzzy contra continuous, Advances in Fuzzy System, **1**(2006), 35-44.

[11] J. H. Park and J. K. Park, Hausdorffness on generalized intuitionistic fuzzy filter, Information Science, **168**(2004), 95-110.

[12] A. A. Ramadan, S. E. Abbas and A. A. Abd el-latif, Compactness in Intuitionistic fuzzy topological spaces, International Journal of Mathematics and Mathematical Science, **1**(2005), 19-32.

[13] R. Santhi and D. Jayathi, Intuitionistic Fuzzy Contra generalized semi pre continuous mapping, Advances in Fuzzy Mathematics, **5**(2010), No. 2, 173-186.

[14] Smets, P. The degree of belief in a fuzzu event, Information sciences, **25**(1981), 1-19.

[15] Sugeno, M. An Introductory survey of fuzzy control, Information sciences, **36**(1985), 59-83.

[16] L. A. Zadeh, Fuzzy sets, Inform. and Control, **8**(1965), 338-353.

Scientia Magna
Vol. 7 (2011), No. 4, 19-22

On the asymptotic properties of triangular base sequence

Le Huan

Department of Mathematics, Northwest University,
Xi'an, Shaanxi, P. R. China
E-mail: huanle1017@163.com

Abstract Let $\{T_n\} = \{1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, 1010, 1011, 10000, 10001, 10002, 10010, 10011, 10012, 100000, 100001, 100002, 100010, 100011, 100012, 100100, 1000000, 1000001, 1000002, 1000010, 1000011, 1000012, 1000100, \cdots\}$ denotes the triangular base sequence. The main purpose of this paper is using the elementary method and the properties of the geometric progression to study the convergent properties of the series consists of reciprocal elements of the set and the asymptotic properties of the triangular base sequence, then proved that the series $\sum_{n=1}^{\infty} \frac{1}{T_n}$ is convergent and obtained an asymptotic formula for $\ln(T_n)$.

Keywords Triangular base, convergent properties, asymptotic formula, elementary method.

§1. Introduction and main results

American-Romanian number theorist F. Smarandache introduced hundreds of interesting sequences and unsloved problems in reference [1] and [2], many authors had studied these problems before and obtained some interesting results, see reference [3] and [4]. In [1], he denotes triangular base sequence: $\{T_n\} = \{1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, 1010, 1011, 10000, 10001, 10002, 10010, 10011, 10012, 100000, 100001, 100002, 100010, 100011, 100012, 100100, 1000000, 1000001, 1000002, 1000010, 1000011, 1000012, 1000100, \ldots\}$. In a general case: if we want to design a base such that any number

$$A = \overline{(a_n \cdots a_2 a_1)}_B \triangleq \sum_{i=1}^{n} a_i b_i,$$

with all $a_i = 0, 1, \cdots, t_i$ for $i \geq 1$, where $t_i \geq 1$, then the base should be $b_1 = 1$, $b_{i+1} = (t_i+1)*b_i$ for $i \geq 1$.

Numbers written in the triangular base defined as follows: $t(n) = \frac{n(n+1)}{2}$, $n \geq 1$. So we

can easily get the first several terms:

$$\begin{aligned}
1 &= 1\cdot\frac{1(1+1)}{2};\\
2 &= 2\cdot\frac{1(1+1)}{2};\\
3 &= 1\cdot\frac{2(2+1)}{2}+0\cdot\frac{1(1+1)}{2};\\
&\vdots\\
8 &= 1\cdot\frac{3(3+1)}{2}+0\cdot\frac{2(2+1)}{2}+2\cdot\frac{1(1+1)}{2};\\
&\vdots
\end{aligned}$$

So in this sequence $T_1=\overline{a_1}=1$, $T_2=\overline{a_1}=2$, $T_3=\overline{a_2a_1}=10,\cdots,T_8=\overline{a_3a_2a_1}=102,\cdots$. we note that in the triangular base sequence a_i can only to be 0, 1, 2 and $a_n\neq 0$.

There have no people studied the properties of this sequence until now, at least we couldn't find any reference about it. In this paper, we first study the properties of this sequence and obtain substantive progress. That is, we have proved these two results:

Theorem 1.1. The series of positive terms which are formed by triangular base sequence

$$S=\sum_{n=1}^{\infty}\frac{1}{T_n}$$

is convergent and $S<2$.

Theorem 1.2. Let $\{T_n\}$ denotes the triangular base sequence, then we have the asymptotic formula

$$\ln(T_n)=\sqrt{2n}\cdot\ln 10+O(1).$$

§2. Proof of the theorems

In this section, we shall use the elementary method and the structure of triangular base sequence to complete the proof of our theorems. The elementary methods used in this paper can be found in reference [5] and [6], we don't repeat here.

First we prove Theorem 1.1. We will classify the sequence $\{T_n\}$ by the digit of each element: there are two elements 1, 2 which contain one digit in the decimal; There are three elements 10, 11, 12 which contain two digits in the decimal; There are four elements 100, 101, 102, 110 which contain three digits in the decimal$\cdots$; Generally, there are $k+1$ elements which contain k digits in the decimal. Let $u\in\{T_n\}$ and the digit of u is i in the decimal, then we have:

$$10^{i-1}=1\overbrace{00\cdots00}^{i-1}\leq u\leq\overbrace{22\cdots22}^{i}=2\cdot\frac{10^i-1}{9}. \tag{1}$$

Obviously, in the interval of $\frac{k(k+1)}{2}-1<n\leq\frac{(k+1)(k+2)}{2}-1$, the digit of T_n is k in the decimal, while the total of T_n which satisfy the digit is k in the decimal is $k+1$. Then from (1) we can

immediately deduce the estimation formula

$$S=\sum_{n=1}^{\infty}\frac{1}{T_n}=\sum_{k=1}^{\infty}\sum_{\frac{k(k+1)}{2}-1<n\leq\frac{(k+1)(k+2)}{2}-1}\frac{1}{T_n}<1+\frac{1}{2}+\sum_{k=2}^{\infty}\frac{k+1}{10^{k-1}}. \tag{2}$$

Now using the property of geometric series to estimate $S^{'}=\sum\limits_{k=2}^{\infty}\frac{k+1}{10^{k-1}}$,

$$\begin{aligned}S^{'} &= \sum_{k=2}^{\infty}\frac{k+1}{10^{k-1}}\\ &= \frac{3}{10}+\frac{4}{10^2}+\frac{5}{10^3}+\frac{6}{10^4}+\cdots\\ \frac{1}{10}S^{'} &= \frac{1}{10}\sum_{k=2}^{\infty}\frac{k+1}{10^{k-1}}\\ &= \frac{3}{10^2}+\frac{4}{10^3}+\frac{5}{10^4}+\frac{6}{10^5}+\cdots\end{aligned}$$

Make difference of the following formula we have

$$\begin{aligned}(1-\frac{1}{10})S^{'} &= \frac{3}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\frac{1}{10^5}+\cdots\\ &= \frac{2}{10}+\frac{\frac{1}{10}}{1-\frac{1}{10}}.\end{aligned}$$

Thus

$$S^{'}=\frac{2}{9}+\frac{10}{81}<\frac{1}{2}. \tag{3}$$

Combine (2) and (3) we get

$$S=\sum_{n=1}^{\infty}\frac{1}{T_n}<1+\frac{1}{2}+\frac{1}{2}=2. \tag{4}$$

From (4) and the convergence rule of series of positive terms we can immediately deduce that $\sum\limits_{n=1}^{\infty}\frac{1}{T_n}$ is convergent, so we proved Theorem 1.1.

Now we prove Theorem 1.2. For any real integer n, there insists real integer m such that $\frac{m(m+1)}{2}-1<n\leq\frac{(m+1)(m+2)}{2}-1$, when n satisfy $\frac{m(m+1)}{2}-1<n\leq\frac{(m+1)(m+2)}{2}-1$, the digit of T_n is m in decimal scale .

So from (1) we know that T_n must satisfy the inequality

$$10^{m-1}=1\overbrace{00\cdots00}^{m-1}\leq T_n\leq\overbrace{22\cdots22}^{m}=2\cdot\frac{10^m-1}{9}. \tag{5}$$

Taking the logarithm of (5) in two sides we get

$$\ln(T_n)=m\cdot\ln 10+O(1). \tag{6}$$

On the other hand, by the definition of m, $\frac{m(m+1)}{2}-1<n\leq\frac{(m+1)(m+2)}{2}-1$, we have estimation formula

$$m=\sqrt{2n}+O(1).$$

Combining (5) and (6) we obtain the asymptotic formula

$$\ln(T_n) = \sqrt{2n} \cdot \ln 10 + O(1).$$

This completes the proof of our Theorems.

References

[1] F. Smarandache, Sequences of numbers involved in unsolved problems, Phoenix, Hexis, 2006, 33.

[2] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[3] Krassimir T. Atanassov, On some of the Smarandache's problem, American Research Press, Lupton, AZ USA, 1999.

[4] Krassimir T. Atanassov, Some Smarandache problems, Hexis, Phoenix, 2004.

[5] T. M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[6] Zhang Wenpeng, The elementary number theory (in Chinese), Shaanxi Normal University Press, Xi'an, 2007.

Scientia Magna
Vol. 7 (2011), No. 4, 23-28

A new class of double interval numbers

Ayhan Esi

Department of Mathematics, Science and Art Faculty, Adiyaman University,
02040, Adiyaman, Turkey
E-mail: aesi23@hotmail.com

Abstract In this paper we introduce the new concept of double interval valued sequence space ${}_2\overline{\ell}(p)$, where $p=(p_{k,l})$ is a double sequence of bounded strictly positive numbers. We study its different properties like completeness, solidness, converge free, symmetricity etc. We prove some inclusion relations also.

Keywords Completeness, interval numbers.
2000 Mathematics Subject Classification: 40C05, 40J05, 46A45.

§1. Introduction and preliminaries

Interval arithmetic was first suggested by Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [9] in 1959 and Moore and Yang [10] 1962. Furthermore, Moore and others [2],[3],[6],[10] have developed applications to differential equations.

Chiao in [1] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [11] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Later Esi in [5] introduced and studied strongly almost $\lambda-$convergence and statistically almost $\lambda-$convergence of interval numbers. Recently Esi [4] introduced and studied the new concept of interval valued sequence space $\overline{\ell}(p)$, where $p=(p_k)$ is a sequence of bounded strictly positive numbers.

A set consisting of a closed interval of real numbers x such that $a\leq x\leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $I\mathbb{R}$. Any elements of $I\mathbb{R}$ is called closed interval and denoted by $\overline{x}$. That is $\overline{x}=\{x\in\mathbb{R}:\ a\leq x\leq b\}$. An interval number $\overline{x}$ is a closed subset of real numbers [1]. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x}_1,\ \overline{x}_2\in I\mathbb{R}$, we have $\overline{x}_1=\overline{x}_2\Leftrightarrow x_{1_l}{=}x_{2_l},\ x_{1_r}{=}x_{2_r}$. $\overline{x}_1+\overline{x}_2=\{x\in\mathbb{R}:\ x_{1_l}+x_{2_l}\leq x\leq x_{1_r}+x_{2_r}\}$, and if $\alpha\geq 0$, then $\alpha\overline{x}=\{x\in\mathbb{R}:\ \alpha x_{1_l}\leq x\leq\alpha x_{1_r}\}$ and if $\alpha<0$, then $\alpha\overline{x}=\{x\in\mathbb{R}:\ \alpha x_{1_r}\leq x\leq\alpha x_{1_l}\}$,

$\overline{x}_1.\ \overline{x}_2=$

$\{x\in\mathbb{R}:\min\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\}\leq x\leq\max\{x_{1_l}x_{2_l},\ x_{1_l}x_{2_r},\ x_{1_r}x_{2_l},\ x_{1_r}x_{2_r}\}\}.$

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d\left(\overline{x}_1,\ \overline{x}_2\right)=\max\left\{\left|x_{1_l}-x_{2_l}\right|,\ \left|x_{1_r}-x_{2_r}\right|\right\}.^{[8]}$$

In the special case $\overline{x}_1=[a,\ a]$ and $\overline{x}_2=[b,\ b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f:\mathbb{N}\to\mathbb{R}$ by $k\to f(k)=\overline{x}$, $\overline{x}=(\overline{x}_k)$. Then $\overline{x}=(\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $\overline{x}=(\overline{x}_k)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [7].

In the reference [1] definition of convergence of interval numbers is defined as follows:

A sequence $\overline{x}=(\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon>0$ there exists a positive integer k_o such that $d\left(\overline{x}_k,\ \overline{x}_o\right)<\varepsilon$ for all $k\geq k_o$ and we denote it by $\lim_k\overline{x}_k=\overline{x}_o$.

Thus, $\lim_k\overline{x}_k=\overline{x}_o\Leftrightarrow\lim_k x_{k_l}=x_{o_l}$ and $\lim_k x_{k_r}=x_{o_r}$.

Now we give new definitions for interval sequences as follows:

An interval valued double sequence space ${}_2\overline{E}$ is said to be solid if $\overline{y}=\left(\overline{y}_{k,l}\right)\in{}_2\overline{E}$ whenever $\left|\overline{y}_{k,l}\right|\leq\left|\overline{x}_{k,l}\right|$ for all $k,\ l\in\mathbb{N}$ and $\overline{x}=(\overline{x}_{k,l})\in{}_2\overline{E}$.

An interval valued double sequence space ${}_2\overline{E}$ is said to be monotone if ${}_2\overline{E}$ contains the canonical pre-image of all its step spaces.

An interval valued double sequence space ${}_2\overline{E}$ is said to be convergence free if $\overline{y}=\left(\overline{y}_{k,l}\right)\in{}_2\overline{E}$ whenever $\overline{x}=(\overline{x}_{k,l})\in{}_2\overline{E}$ and $\overline{x}_{k,l}=\overline{0}$ implies $\overline{y}_{k,l}=\overline{0}$.

An interval valued double sequence space ${}_2\overline{E}$ is said to be symmetric if ${}_2S\left(\overline{x}\right)\subset{}_2\overline{E}$ for all $\overline{x}=(\overline{x}_{k,l})\in{}_2\overline{E}$, where ${}_2S\left(\overline{x}\right)$ denotes the set of all permutations of the elements of $\overline{x}=(\overline{x}_{k,l})$.

Throughout the paper, $p=(p_{k,l})$ is a double sequence of bounded strictly positive numbers. We define the following interval valued double sequence space:

$${}_2\overline{\ell}\left(p\right)=\left\{\overline{x}=(\overline{x}_{k,l}):\ \sum_{k,l=1,1}^{\infty,\infty}\left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}}<\infty\right\}$$

and if $p_{k,l}=1$ for all $k,\ l\in\mathbb{N}$, then we have

$${}_2\overline{\ell}=\left\{\overline{x}=(\overline{x}_{k,l}):\ \sum_{k,l=1,1}^{\infty,\infty}d\left(\overline{x}_{k,l},\ \overline{0}\right)<\infty\right\}.$$

Lemma 1.1. If a double sequence space ${}_2\overline{E}$ is solid, then it is monotone.

§2. Main results

Theorem 2.1. The double sequence space ${}_2\overline{\ell}\left(p\right)$ is a complete metric space with respect to the metric ρ defined by

$$\rho\left(\overline{x},\ \overline{y}\right)=\left[\sum_{k,l=1,1}^{\infty,\infty}\left[d\left(\overline{x}_{k,l},\ \overline{y}_{k,l}\right)\right]^{p_{k,l}}\right]^{\frac{1}{M}},$$

where $M = \max\left(1,\ \sup_{k,l} p_{k,l}\right)$.

Proof. Let $\left(\overline{x}^i\right)$ be a Cauchy sequence in ${}_2\overline{\ell}\,(p)$. Then for a given $\varepsilon > 0$, there exists $n_o \in \mathbb{N}$, such that

$$\rho\left(\overline{x}^i,\ \overline{x}^j\right) < \varepsilon, \text{ for all } \ i,\ j \geq n_o.$$

Then

$$\begin{aligned}
&\left[\sum_{k,l=1,1}^{\infty,\infty}\left[d\left(\overline{x}^i_{k,l},\ \overline{y}^j_{k,l}\right)\right]^{p_{k,l}}\right]^{\frac{1}{M}} < \varepsilon, \text{ for all } \ i,\ j \geq n_o \\
\Rightarrow &\sum_{k,l=1,1}^{\infty,\infty}\left[d\left(\overline{x}^i_{k,l},\ \overline{y}^j_{k,l}\right)\right]^{p_{k,l}} < \varepsilon^M, \text{ for all } \ i,\ j \geq n_o \\
\Rightarrow &\, d\left(\overline{x}^i_{k,l},\ \overline{y}^j_{k,l}\right) < \varepsilon, \text{ for all } \ i,\ j \geq n_o \text{ and all } \ k,\ l \in \mathbb{N}. \qquad (1)
\end{aligned}$$

This means that $\left(\overline{x}^i_{k,l}\right)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a Banach space, $\left(\overline{x}^i_{k,l}\right)$ is convergent. Now, let

$$\lim_{i\to\infty} \overline{x}^i_{k,l} = \overline{x}_{k,l} \ \text{ for each } k,\ l \in \mathbb{N} \text{ and } \overline{x} = \left(\overline{x}_{k,l}\right).$$

Taking limit as $j \to \infty$ in (1), we have

$$\rho\left(\overline{x}^i,\ \overline{x}\right) < \varepsilon, \text{ for all } \ i \geq n_o.$$

Now for all $i \geq n_o$

$$\rho\left(\overline{x},\ \overline{0}\right) \leq \rho\left(\overline{x},\ \overline{x}^i\right) + \rho\left(\overline{x}^i,\ \overline{0}\right) < \infty.$$

Thus $\overline{x} = \left(\overline{x}_{k,l}\right) \in_2 \overline{\ell}\,(p)$ and so ${}_2\overline{\ell}\,(p)$ is complete. This completes the proof.

Theorem 2.2. The sequence space ${}_2\overline{\ell}\,(p)$ is solid as well as monotone.

Proof. Let $\overline{x} = \left(\overline{x}_{k,l}\right) \in_2 \overline{\ell}\,(p)$ and $\overline{y} = \left(\overline{y}_{k,l}\right)$ be a interval valued sequence such that $\left|\overline{y}_{k,l}\right| \leq \left|\overline{x}_{k,l}\right|$ for all $k,\ l \in \mathbb{N}$. Then

$$\sum_{k,l=1,1}^{\infty,\infty}\left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} < \infty$$

and

$$\sum_{k,l=1,1}^{\infty,\infty}\left[d\left(\overline{y}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} \leq \sum_{k,l=1,1}^{\infty,\infty}\left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} < \infty.$$

Thus $\overline{y} = \left(\overline{y}_{k,l}\right) \in_2 \overline{\ell}\,(p)$ and so ${}_2\overline{\ell}\,(p)$ is solid. Also by Lemma 1.1, it follows that the space ${}_2\overline{\ell}\,(p)$ is monotone. This completes the proof.

Theorem 2.3. The double sequence space $\overline{\ell}\,(p)$ is not convergence free in general.

Proof. The result follows from the following example.

Example 2.1. Let

$$p_{k,l} = \begin{cases} 2, & \text{if } k = l; \\ \frac{1}{kl}, & \text{if } k \neq l. \end{cases}$$

We consider the double interval sequence $\overline{x} = (\overline{x}_{k,l})$ as follows:

$$\overline{x}_{k,l} = \begin{cases} \left[-\frac{1}{k^2},\ 0\right], & \text{if } k = l; \\ [0,\ 0], & \text{if } k \neq l. \end{cases}.$$

Then

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} = \sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)^2 < \infty,$$

so $\overline{x} = (\overline{x}_{k,l}) \in_2 \overline{\ell}(p)$. Now let us consider the double interval sequence $\overline{y} = \left(\overline{y}_{k,l}\right)$ defined as follows:

$$\overline{y}_{k,l} = \begin{cases} \left[-k^2,\ k^2\right], & \text{if } k = l; \\ [0,\ 0], & \text{if } k \neq l. \end{cases}.$$

Then

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{y}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} = \sum_{k,l=1,1}^{\infty,\infty} \left(k^2\right)^2 = \infty,$$

so $\overline{y} = \left(\overline{y}_{k,l}\right) \notin_2 \overline{\ell}(p)$. Hence ${}_2\overline{\ell}(p)$ is not convergence free. This completes the proof.

Theorem 2.4. The double sequence space ${}_2\overline{\ell}(p)$ is not symmetric.

Proof. The result follows from the following example.

Example 2.2. Let

$$p_{k,l} = \begin{cases} 2, & \text{if } k = l; \\ 1, & \text{if } k \neq l. \end{cases}.$$

We consider the double interval sequence $\overline{x} = (\overline{x}_{k,l})$ as follows:

$$\overline{x}_{k,l} = \begin{cases} \left[-\frac{1}{\sqrt{2}k},\ 0\right], & \text{if } k = l; \\ [0,\ 0], & \text{if } k \neq l. \end{cases}.$$

Then

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}k}\right)^2 < \infty,$$

so $\overline{x} = (\overline{x}_{k,l}) \in_2 \overline{\ell}(p)$. Now let us consider the rearrangement the double interval sequence $\overline{y} = \left(\overline{y}_{k,l}\right)$ of $\overline{x} = (\overline{x}_{k,l})$ defined as follows:

$$(\overline{y_{k,l}}) = \begin{pmatrix} \overline{x}_{1,1} & \overline{x}_{1,2} & \overline{x}_{1,3} & \cdots \\ \overline{x}_{2,2} & \overline{x}_{2,1} & \overline{x}_{2,3} & \cdots \\ \overline{x}_{3,3} & \overline{x}_{3,2} & \overline{x}_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{y}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} = \left(\frac{1}{\sqrt{2}k}\right)^2 + \sum_{k=1}^{\infty} \frac{1}{\sqrt{2}k} = \infty.$$

Hence $\overline{y} = \left(\overline{y}_{k,l}\right) \notin_2 \overline{\ell}\,(p)$ and so the double sequence space ${}_2\overline{\ell}\,(p)$ is not symmetric. This completes the proof.

Theorem 2.5. (a) If $0 < p_{k,l} \leq q_{k,l} < 1$ for all $k,\ l \in \mathbb{N}$, then ${}_2\overline{\ell}\,(p) \subset_2 \overline{\ell}\,(q)$ and the inclusion is proper.

(b) If $0 < \inf_{k,l} p_{k,l} \leq p_{k,l}$ for all $k,\ l \in \mathbb{N}$, then ${}_2\overline{\ell}\,(p) \subset_2 \overline{\ell}$ and if $1 < p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty$ for all $k,\ l \in \mathbb{N}$, then ${}_2\overline{\ell} \subset_2 \overline{\ell}\,(p)$.

Proof. (a) Let $\overline{x} = (\overline{x}_{k,l}) \in_2 \overline{\ell}\,(p)$. Then

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} < \infty.$$

So there exists $n_o \in \mathbb{N}$ such that

$$\left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} < 1 \text{ for all } \ k,\ l \geq n_o.$$

Thus

$$\left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{q_{k,l}} < \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} \text{ for all } \ k,\ l \geq n_o$$

and so,

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{q_{k,l}} < \infty.$$

Thus $\overline{x} = (\overline{x}_{k,l}) \in_2 \overline{\ell}\,(q)$. The inclusion is strict and it follows from the following example.

Example 2.3. We consider the double interval sequence $\overline{x} = (\overline{x}_{k,l})$ as follows:

$$\overline{x}_{k,l} = \begin{cases} \left[-\frac{1}{\sqrt{3k}},\ 0\right], & \text{if } k = l; \\ [0,\ 0], & \text{if } k \neq l. \end{cases}$$

and

$$q_{k,l} = \begin{cases} 2 + \frac{1}{k}, & \text{if } k = l; \\ 3, & \text{if } k \neq l. \end{cases} \quad ; \quad p_k = \begin{cases} 1, & \text{if } k = l; \\ 2, & \text{if } k \neq l. \end{cases} .$$

Then

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{q_{k,l}} = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{3k}}\right)^{2+\frac{1}{k}} < \infty$$

and so $\overline{x} = (\overline{x}_{k,l}) \in_2 \overline{\ell}\,(q)$, but

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{3k}} = \infty,$$

so $\overline{x} = (\overline{x}_{k,l}) \notin_2 \overline{\ell}\,(p)$. This completes the proof.

(b) The first part of the result follows from the inequality

$$\sum_{k,l=1,1}^{\infty,\infty} d\left(\overline{x}_{k,l},\ \overline{0}\right) \leq \sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}}$$

and the second part of the result follows from the inequality

$$\sum_{k,l=1,1}^{\infty,\infty} \left[d\left(\overline{x}_{k,l},\ \overline{0}\right)\right]^{p_{k,l}} \leq \sum_{k,l=1,1}^{\infty,\infty} d\left(\overline{x}_{k,l},\ \overline{0}\right).$$

References

[1] Kuo-Ping Chiao, Fundamental properties of interval vector max-norm, Tamsui Oxf. J. Math. Sci., **18**(2002), No. 2, 219-233.

[2] P. S. Dwyer, Linear Computation, New York, Wiley, 1951.

[3] P. S. Dwyer, Errors of matrix computation, simultaneous equations and eigenvalues, National Bureu of Standards, Applied Mathematics Series, **29**(1953), 49-58.

[4] A. Esi, A new class of interval numbers, Journal of Qafqaz University, Mathematics and Computer Science, **31**(2011), 98-102.

[5] A. Esi, Strongly almost $\lambda-$convergence and statistically almost $\lambda-$convergence of interval numbers, Sci. Magna, **7**(2011), No. 2, 117-122.

[6] P. S. Fischer, Automatic propagated and round-off error analysis, paper presented at the 13th National Meeting of the Association of Computing Machinery, **6**(1958).

[7] S. Markov, Quasilinear spaces and their relation to vector spaces, Electronic Journal on Mathematics of Computation, **2**(2005), No. 1, 1-21.

[8] R. E. Moore and C. T. Yang, Theory of an interval algebra and its application to numeric analysis, RAAG Memories II, Gaukutsu Bunken Fukeyu-kai, Tokyo, 1958.

[9] R. E. Moore, Automatic Error Analysis in Digital Computation, LSMD-48421, Lockheed Missiles and Space Company, 1959.

[10] R. E. Moore and C. T. Yang, Interval Analysis I, LMSD-285875, Lockheed Missiles and Space Company, 1962.

[11] M. Şengönül and A. Eryılmaz, On the sequence spaces of interval numbers, Thai J. Math., **8**(2010), No. 3, 503-510.

Scientia Magna
Vol. 7 (2011), No. 4, 29-34

Linear operators preserving commuting pairs of matrices over semirings [1]

Miaomiao Ren[†], Yizhi Chen[‡] and Guanwei Qin[♯]

Department of Mathematics, Northwest University, Xi'an,
Shaanxi, 710127, P. R. China
E-mail: mralgebra@sohu.com yizhichen1980@126.com qinguanwei11@163.com

Abstract In this paper we characterize linear operators φ on the matrices over the direct product of copies (need not be finite) of a binary Boolean algebra such that φ preserves commuting pairs of matrices.

Keywords Linear operator, binary boolean algebra, commuting pair of matrices, direct pro -duct.

§1. Introduction and preliminaries

A semiring means an algebra $\langle \mathbb{S},\ +,\ \cdot,\ 0,\ 1\rangle$, where $+$ and $\cdot$ are binary, 0 and 1 are nullary, satisfying the following conditions:

(1) $\langle \mathbb{S},\ +,\ 0\rangle$ is a commutative monoid;

(2) $\langle \mathbb{S},\ \cdot,\ 1\rangle$ is a monoid;

(3) $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(a+b)\cdot c=a\cdot c+b\cdot c$, $\forall a,\ b,\ c\ \epsilon\ S$;

(4) $a\cdot 0=0\cdot a=0$, $\forall a\ \epsilon\ S$;

(5) $0\neq 1$.

Let $\mathbb{S}$ be a semiring and $\mathbb{M}_n(\mathbb{S})$ the set of all $n\times n$ matrices over $\mathbb{S}$. I is the identity matrix, and O is the zero matrix. Define $+$ and $\cdot$ on $\mathbb{M}_n(\mathbb{S})$ as follows:

$$A+B=[a_{ij}+b_{ij}]_{n\times n},\quad A\cdot B=[\sum_{k=1}^{n}a_{ik}b_{kj}]_{n\times n}.$$

It is easy to verify that $\langle \mathbb{M}_n(\mathbb{S}),\ +,\ \cdot,\ O,\ I\rangle$ is a semiring with the above operations. Let φ be an operator on $\mathbb{M}_n(\mathbb{S})$. Then we say that φ preserves commuting pairs of matrices if $\varphi(A)\varphi(B)=\varphi(B)\varphi(A)$ whenever $AB=BA$.

It is well known that commutativity of matrices is very important in the theory of matrices. Many authors have studied linear operators that preserve commuting pairs of matrices over fields and semirings. Song and Beasley [9] gave characterizations of linear operators that preserve commuting pairs of matrices over the nonnegative reals. Watkins [14] considered the same

[1]This work is supported by a grant of the science foundation of Northwest University (NC0925) and a grant of the science foundation of Shaanxi Province (2011JQ1017).

problem on matrices over the field of characteristic 0. Moreover, there are papers on linear operators that preserve commuting pairs of matrices over some other special fields (see [1, 2, 8]). In [10], Song and Kang characterized such linear operators over general Boolean algebras and chain semirings.

Linear preserver problem (LPP for short) is one of the most active research areas in matrix theory. It concerns the classification of linear operators that preserve certain functions, relations, subsets, etc., invariant. Although the linear operators concerned are mostly linear operators on matrix spaces over fields or rings, the same problem has been extended to matrices over various semirings (see [3, 5, 6, 7, 9, 10, 13] and references therein).

In the last decades, there are a series of literature on linear operators that preserve invariants of matrices over a given semiring. Idempotent preservers were investigated by Song, Kang and Beasley [11], Dolžan and Oblak [3] and Orel [6]. Nilpotent preservers were discussed by Song, Kang and Jun [13] and Li and Tan [5]. Regularity preservers were studied by Song, Kang, Jun, Beasley and Sze in [4] and [12]. Pshenitsyna [7] considered invertibility preservers. Song and Kang [10] studied commuting pairs of matrices preservers.

In this paper we study linear operators φ on the matrices over the direct product of copies (need not be finite) of a binary Boolean algebra such that φ preserves commuting pairs of matrices.

For convenience, we use $\mathbb{Z}^+$ to denote the set of all positive integers.

Hereafter, $\mathbb{S}$ will always denote an arbitrary semiring unless otherwise specified.

Let A be a matrix in $\mathbb{M}_n(\mathbb{S})$. We denote entry of A in the ith row and jth column by a_{ij} and transpose of A by A^t.

For any $A \in \mathbb{M}_n(\mathbb{S})$ and any $\lambda \in \mathbb{S}$, define

$$\lambda A = [\lambda a_{ij}]_{n\times n}, \quad A\lambda = [a_{ij}\lambda]_{n\times n}.$$

A mapping $\varphi:\ \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ is called a linear operator (see [9]) if

$$\varphi(aA + Bb) = a\varphi(A) + \varphi(B)b$$

for all $a,\ b \in \mathbb{S}$ and all $A,\ B \in \mathbb{M}_n(\mathbb{S})$.

Given $k \in \mathbb{Z}^+$. Let $\mathbb{B}_k$ be the power set of a k-element set $\mathbb{S}_k$ and $\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_k$ the singleton subsets of $\mathbb{S}_k$. We denote ϕ by 0 and $\mathbb{S}_k$ by 1. Define $+$ and $\cdot$ on $\mathbb{M}_n(\mathbb{S})$ by

$$A + B = A\bigcup B, \qquad A \cdot B = A\bigcap B.$$

Then $\langle \mathbb{B}_k,\ +,\ \cdot,\ 0,\ 1\rangle$ becomes a semiring and is called a general Boolean algebra (see [13]). In particular, if $k = 1$ then $\mathbb{B}_1$ is called the binary Boolean algebra (see [13]).

A matrix $A \in \mathbb{M}_n(\mathbb{S})$ is said to be invertible (see [10]) if there exists $B \in \mathbb{M}_n(\mathbb{S})$ such that $AB = BA = I$. A matrix $P \in \mathbb{M}_n(\mathbb{S})$ is called a permutation matrix (see [3]) if it has exactly one entry 1 in each row and each column and 0's elsewhere. If $P \in \mathbb{M}_n(\mathbb{S})$ is a permutation matrix, then $PP^t = P^tP = I$. Note that the only invertible matrices in $\mathbb{M}_n(\mathbb{B}_1)$ are permutation matrices (see [10]).

§2. Main results

Let $(\mathbb{S}_\lambda)_{\lambda\in\wedge}$ be a family of semirings and $\mathbb{S} = \prod_{\lambda\in\wedge}\mathbb{S}_\lambda$. For any $\lambda \in \wedge$ and any $a \in \mathbb{S}$, we use a_λ to denote $a(\lambda)$. Define

$$(a+b)_\lambda = a_\lambda + b_\lambda, \quad (ab)_\lambda = a_\lambda b_\lambda \quad (a,\ b \in \mathbb{S},\ \lambda \in \wedge).$$

It is routine to check that $\langle \mathbb{S},\ +,\ \cdot,\ 0,\ 1\rangle$ is a semiring under the above operations. For any $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{S})$ and any $\lambda \in \wedge$, $A_\lambda := [(a_{ij})_\lambda] \in \mathbb{M}_n(\mathbb{S}_\lambda)$. It is obvious that

$$(A+B)_\lambda = A_\lambda + B_\lambda, \quad (AB)_\lambda = A_\lambda B_\lambda \text{ and } (aA)_\lambda = a_\lambda A_\lambda$$

for all $A,\ B \in \mathbb{M}_n(\mathbb{S})$ and all $a \in \mathbb{S}$.

Hereafter, $\mathbb{S} = \prod_{\lambda\in\wedge}\mathbb{S}_\lambda$, where $\mathbb{S}_\lambda$ is a semiring for any $\lambda \in \wedge$. In the following, we can easily obtain

Lemma 2.1. Let A and B be matrices in $\mathbb{M}_n(\mathbb{S})$. Then the following statements hold:

(i) $A = B$ if and only if $A_\lambda = B_\lambda$ for all $\lambda \in \wedge$;

(ii) $AB = BA$ if and only if $A_\lambda B_\lambda = B_\lambda A_\lambda$ for all $\lambda \in \wedge$;

(iii) A is invertible if and only if A_λ is invertible for all $\lambda \in \wedge$.

The following result is due to Orel [6].

Lemma 2.2. If $\varphi :\ \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ is a linear operator, then for any $\lambda \in \wedge$, there exists a unique linear operator $\varphi_\lambda :\ \mathbb{M}_n(\mathbb{S}_\lambda) \to \mathbb{M}_n(\mathbb{S}_\lambda)$ such that $(\varphi(A))_\lambda = \varphi_\lambda(A_\lambda)$ for all $A \in \mathbb{M}_n(\mathbb{S})$.

Now, let φ a linear operator on $\mathbb{M}_n(\mathbb{S})$. Suppose that φ_λ preserves commuting pairs of matrices for all $\lambda \in \wedge$. For any $A,\ B \in \mathbb{S}$, if $AB = BA$, then for any $\lambda \in \wedge$, $A_\lambda B_\lambda = B_\lambda A_\lambda$. This implies that

$$\varphi_\lambda(A_\lambda)\varphi_\lambda(B_\lambda) = \varphi_\lambda(B_\lambda)\varphi_\lambda(A_\lambda).$$

It follows that

$$\begin{aligned}(\varphi(A)\varphi(B))_\lambda &= (\varphi(A))_\lambda(\varphi(B))_\lambda = \varphi_\lambda(A_\lambda)\varphi_\lambda(B_\lambda)\\ &= \varphi_\lambda(B_\lambda)\varphi_\lambda(A_\lambda) = (\varphi(B))_\lambda(\varphi(A))_\lambda = (\varphi(B)\varphi(A))_\lambda.\end{aligned}$$

Further, $\varphi(A)\varphi(B) = \varphi(B)\varphi(A)$. Thus φ preserves commuting pairs of matrices.

Conversely, assume that φ preserves commuting pairs of matrices. For any $\lambda \in \wedge$ and $A,\ B \in \mathbb{M}_n(\mathbb{S}_\lambda)$, there exist $X,\ Y \in \mathbb{M}_n(\mathbb{S})$ such that $X_\lambda = A,\ Y_\lambda = B$ and $X_\mu = Y_\mu = O$ for any $\mu \neq \lambda$. If $AB = BA$ then $XY = YX$. Since φ preserves commuting pairs of matrices, we have $\varphi(X)\varphi(Y) = \varphi(Y)\varphi(X)$. It follows that $\varphi_\lambda(A)\varphi_\lambda(B) = (\varphi(X))_\lambda(\varphi(Y))_\lambda = (\varphi(X)\varphi(Y))_\lambda = (\varphi(Y)\varphi(X))_\lambda = (\varphi(Y))_\lambda(\varphi(X))_\lambda = \varphi_\lambda(B)\varphi_\lambda(A)$. Hence φ_λ preserves commuting pairs of matrices. In fact, we have proved

Lemma 2.3. Let $\varphi :\ \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ be a linear operator. Then φ preserves commuting pairs of matrices if and only if φ_λ preserves commuting pairs of matrices for all $\lambda \in \wedge$.

Let φ be a linear operator on $\mathbb{M}_n(\mathbb{S})$. Suppose that φ is invertible. For any $\lambda \in \wedge$ and any $A,\ B \in \mathbb{M}_n(\mathbb{S}_\lambda)$, there exist $\overline{A},\ \overline{B} \in \mathbb{M}_n(\mathbb{S})$ such that $(\overline{A})_\lambda = A,\ (\overline{B})_\lambda = B$, and $(\overline{A})_\mu = (\overline{B})_\mu = O$ for any $\mu \neq \lambda$. If $\varphi_\lambda(A) = \varphi_\lambda(B)$ then

$$(\varphi(\overline{A}))_\lambda = \varphi_\lambda(A) = \varphi_\lambda(B) = (\varphi(\overline{B}))_\lambda.$$

Also,

$$(\varphi(\overline{A}))_\mu = (\varphi(\overline{B}))_\mu = \varphi_\mu(O) = O$$

for any $\mu \neq \lambda$. This shows that $\varphi(\overline{A}) = \varphi(\overline{B})$. Since φ is injective, we have $\overline{A} = \overline{B}$. Further,

$$A = (\overline{A})_\lambda = (\overline{B})_\lambda = B.$$

Thus φ_λ is injective.

On the other hand, since φ is surjective, it follows that there exists $X \in \mathbb{M}_n(\mathbb{S})$ such that $\varphi(X) = \overline{B}$. We can deduce that

$$B = (\overline{B})_\lambda = (\varphi(X))_\lambda = \varphi_\lambda(X_\lambda).$$

That is to say, φ_λ is surjective. Hence φ_λ is invertible. We have therefore established half of

Lemma 2.4. Let φ be a linear operator on $\mathbb{M}_n(\mathbb{S})$. Then φ is invertible if and only if φ_λ is invertible for all $\lambda \in \wedge$.

Proof. We only need to prove the remaining half. Assume that φ_λ is invertible for any $\lambda \in \wedge$. For any $A,\ B \in \mathbb{M}_n(\mathbb{S})$ and any $\lambda \in \wedge$, if $\varphi(A) = \varphi(B)$ then

$$\varphi_\lambda(A_\lambda) = (\varphi(A))_\lambda = (\varphi(B))_\lambda = \varphi_\lambda(B_\lambda).$$

Since φ_λ is injective, we have $A_\lambda = B_\lambda$. By Lemma 2.1(i) it follows that $A = B$. Hence φ is injective. Since φ_λ is surjective, there exists $X^{(\lambda)} \in \mathbb{M}_n(\mathbb{S}_\lambda)$ such that $\varphi_\lambda(X^{(\lambda)}) = B_\lambda$. Let $X \in \mathbb{M}_n(\mathbb{S})$ satisfy $X_\lambda = X^{(\lambda)}$ for any $\lambda \in \wedge$. It is clear that $\varphi(X) = B$, and so φ is surjective. Thus φ is invertible as required.

The following lemma, due to Song and Kang [10], characterize invertible linear operators preserveing commuting pairs of matrices over a binary Boolean algebra.

Lemma 2.5. Let φ be a linear operator on $\mathbb{M}_n(\mathbb{B}_1)$. Then φ is an invertible linear operator that preserves commuting pairs of matrices if and only if there exists a permutation matrix $P \in \mathbb{M}_n(\mathbb{B}_1)$ such that either

(a) $\varphi(X) = PXP^t$ for all $X \in \mathbb{M}_n(\mathbb{B}_1)$, or

(b) $\varphi(X) = PX^tP^t$ for all $X \in \mathbb{M}_n(\mathbb{B}_1)$.

Next, we have

Theorem 2.1. Let $\mathbb{S} = \prod_{\lambda\in\wedge} \mathbb{S}_\lambda$, where $\mathbb{S}_\lambda = \mathbb{B}_1$ for any $\lambda \in \wedge$. Let $\varphi :\ \mathbb{M}_n(\mathbb{S}) \to \mathbb{M}_n(\mathbb{S})$ be a linear operator. Then φ is an invertible linear operator that preserves commuting pairs of matrices if and only if there exist invertible matrix $U \in \mathbb{M}_n(\mathbb{S})$ and $f_1,\ f_2 \in \mathbb{S}$ such that

$$\varphi(X) = U(f_1X + f_2X^t)U^t$$

for all $X \in \mathbb{M}_n(\mathbb{S})$, where $(f_1)_\lambda \neq (f_2)_\lambda$ for any $\lambda \in \wedge$.

Proof. ($\Longrightarrow$) It follows from Lemma 2.3 and Lemma 2.4 that φ_λ is an invertible linear operator which preserves commuting pairs of matrices for any $\lambda \in \wedge$. By Lemma 2.5, there exists $U \in \mathbb{M}_n(\mathbb{S})$ such that either

$$\varphi_\lambda(X) = U_\lambda X U_\lambda^t \tag{1}$$

for all $X \in \mathbb{M}_n(\mathbb{S}_\lambda)$, or

$$\varphi_\lambda(X) = U_\lambda X^t U_\lambda^t \tag{2}$$

for all $X \in \mathbb{M}_n(\mathbb{S}_\lambda)$. Moreover, U_λ is a permutation matrix for any $\lambda \in \wedge$. We have by Lemma 2.1 (iii) that U is invertible. Let $\wedge_i := \{\lambda \in \wedge |\ \varphi_\lambda$ is the form of $(i)\},\ i = 1,\ 2$. It is evident that $\wedge_1 \bigcap \wedge_2 = \phi,\ \wedge_1 \bigcup \wedge_2 = \wedge$. For $i = 1,\ 2$, let $f_i \in \mathbb{S}$ satisfy $(f_i)_\lambda = 1$ if $\lambda \in \wedge_i$ and 0 otherwise. We conclude that $\varphi(X) = U(f_1X + f_2X^t)U^t$ for all $X \in \mathbb{M}_n(\mathbb{S})$ as required.

($\Longleftarrow$) For any $\lambda \in \wedge$ and any $X \in \mathbb{M}_n(\mathbb{S}_\lambda)$, there exists $Y \in \mathbb{M}_n(\mathbb{S})$ such that $X = Y_\lambda$. We have

$$\varphi_\lambda(X) = \varphi_\lambda(Y_\lambda) = (\varphi(Y))_\lambda = (U(f_1Y + f_2Y^t)U^t)_\lambda.$$

If $(f_1)_\lambda = 1,\ (f_2)_\lambda = 0$, then $\varphi_\lambda(X) = U_\lambda X U_\lambda^t$ for all $X \in \mathbb{M}_n(\mathbb{S}_\lambda)$. Otherwise, $\varphi_\lambda(X) = U_\lambda X_\lambda^t U_\lambda^t$ for all $X \in \mathbb{M}_n(\mathbb{S}_\lambda)$. We have by Lemma 2.1 (iii) that U_λ is invertible. It follows from Theorem 2.1 that φ_λ is an invertible linear operator that preserves commuting pairs of matrices. Therefore, φ is an invertible linear operator that preserves commuting pairs of matrices by Lemma 2.3 and Lemma 2.4.

Song and Kang [10] characterize invertible linear operators which preserve commuting pairs of matrices over general Boolean algebra. Recall that a general Boolean algebra is isomorphic to a direct product of binary Boolean algebras. Up to isomorphism, we obtain characterization of invertible linear operators which preserve commuting pairs of matrices over general Boolean algebra.

Example 2.1. Let $\mathbb{S} = \mathbb{B}_1 \times \mathbb{B}_1 \times \mathbb{B}_1$. Take

$$U = \begin{bmatrix} (0,0,1) & (1,0,0) & (0,1,0) \\ (0,1,0) & (0,0,1) & (1,0,0) \\ (1,0,0) & (0,1,0) & (0,0,1) \end{bmatrix}$$

in $\mathbb{M}_3(\mathbb{S})$ and $f_1 = (0,1,0),\ f_2 = (1,0,1)$ in $\mathbb{S}$. Define an operator φ on $\mathbb{M}_3(\mathbb{S})$ by

$$\varphi(X) = U(f_1X + f_2X^t)U^t$$

for all $X \in \mathbb{M}_3(\mathbb{S})$.

It is obvious that $U_\lambda(\lambda = 1,\ 2,\ 3)$ are all permutation matrices. By Theorem 2.1, φ is an invertible linear operator which preserves commuting pairs of matrices.

Example 2.2. Let $\mathbb{S} = \prod_{k \in \mathbb{Z}^+} \mathbb{S}_k$, where $\mathbb{S}_k = \mathbb{B}_1$ for any $k \in \mathbb{Z}^+$. Take $a,\ b \in \mathbb{S}$, where $a_{2k-1} = 1,\ a_{2k} = 0,\ b_{2k-1} = 0$ and $b_{2k} = 1$ for any $k \in \mathbb{Z}^+$. Let

$$U = \begin{bmatrix} b & a & 0 \\ a & 0 & b \\ 0 & b & a \end{bmatrix}$$

be a matrix in $\mathbb{M}_3(\mathbb{S})$. Define an operator φ on $\mathbb{M}_3(\mathbb{S})$ by

$$\varphi(X) = U(aX + bX^t)U^t$$

for all $X \in \mathbb{M}_3(\mathbb{S})$.

We have by Theorem 2.1 that φ is an invertible linear operator which preserves commuting pairs of matrices.

References

[1] L. B. Beasley, Linear transformations on matrices: the invariance of commuting pairs of matrices, **6**(1978/79), No. 3, 179-183.

[2] M. D. Choi, A. A. Jafarian and H. Radjavi, Linear maps preserving commutativity, Linear Algebra Appl., **87**(1976), 227-241.

[3] D. Dolžan and P. Oblak, Idempotent matrices over antirings, Linear Algebra Appl., **431**(2009), No. 5-7, 823-832.

[4] K. T. Kang, S. Z. Song and Y. B. Jun, Linear operators that strongly preserve regularity of fuzzy matrices, Math. Commun., **15**(2010), No. 1, 243-254.

[5] H. H. Li and Y. J. Tan, Linear operators that strongly preserves nilpotent matrices over antinegative semirings, Northeast. Math. J., **23**(2007), No. 1, 71-86.

[6] M. Orel, Nonbijective idempotents preservers over semirings, J. Korean Math. Soc., **47**(2010), No. 4, 805-818.

[7] O. A. Pshenitsyna, Maps preserving invertibility of matrices over semirings, Russ. Math. Surv., **64**(2009), No. 1, 162-164.

[8] H. Radjavi, Commtativity-preserving operators on symmetric matrices, Linear Algebra Appl., **61**(1984), 219-224.

[9] S. Z. Song and L. B. Beasley, Linear operators that preserve commuting pairs of non-negative real matrices, Linear Multilinear Algebra, **51**(2003), No. 3, 279-283.

[10] S. Z. Song and K. T. Kang, Linear maps that preserve commuting pairs of matrices over general Boolean algebra, J. Korean Math. Soc., **43**(2006), No. 1, 77-86.

[11] S. Z. Song, K. T. Kang and L. B. Beasley, Idempotent matrix preservers over Boolean algebras, J. Korean Math. Soc., **44**(2007), No. 1, 169-178.

[12] S. Z. Song, K. T. Kang, L. B. Beasley and N. S. Sze, Regular matrices and their strong preservers over semirings, Linear Algebra Appl., **429**(2008), 209-223.

[13] S. Z. Song, K. T. Kang and Y. B. Jun, Linear preservers of Boolean nilpotent matrices, J. Korean Math. Soc., **43**(2006), No. 3, 539-552.

[14] W. Watkins, Linear maps that preserve commuting pairs of matrices, Linear Algebra Appl., **14**(1976), No. 1, 29-35.

Scientia Magna
Vol. 7 (2011), No. 4, 35-41

Conditions for a subclass of analytic functions

Sukhwinder Singh Billing

Department of Applied Sciences, Baba Banda Singh Bahadur Engineering College,
Fatehgarh Sahib, 140407, Punjab, India
E-mail: ssbilling@gmail.com

Abstract This paper is an application of a lemma due to Miller and Mocanu [1], using which we find certain sufficient conditions in terms of the differential operator

$$\frac{zf''(z)}{f'(z)} + (\mu+1)\left(1 - \frac{zf'(z)}{f(z)}\right)$$

for $f \in \mathcal{A}$ to belong to the class $U(\lambda,\ \mu,\ q(z))$ where

$$U(\lambda,\ \mu,\ q(z)) = \left\{ f \in \mathcal{H} :\ \frac{f(z)}{z} \neq 0 \text{ and } f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec q(z),\ z \in \mathbb{E} \right\},$$

and $q,\ q(z) \neq 0$ is univalent in $\mathbb{E}$ and μ a complex number. As special cases of our main result, we find certain significant results regarding starlikeness and strongly starlikeness which extend some known results.

Keywords Analytic function, differential subordination, starlike function, strongly starlike function.

§1. Introduction and preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z :\ |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions f which are analytic in $\mathbb{E}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$, has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$ in $\mathbb{E}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha,\ 0 < \alpha \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{E},$$

equivalently

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha},\ z \in \mathbb{E}.$$

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\phi :\ \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\phi(p(z),\ zp'(z);\ z) \prec h(z),\ \phi(p(0),\ 0;\ 0) = h(0), \tag{1}$$

then p is called a solution of the first order differential subordination (1). The univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

Let

$$U(\lambda,\ \mu,\ q(z)) = \left\{ f \in \mathcal{H} :\ \frac{f(z)}{z} \neq 0 \text{ and } f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec q(z),\ z \in \mathbb{E} \right\},$$

and $q,\ q(z) \neq 0$ is univalent in $\mathbb{E}$ and μ a complex number. Throughout this paper, value of the complex power taken is the principal one.

Fournier and Ponnusamy [2] studied the class

$$U(\lambda,\ \mu) = \left\{ f \in \mathcal{H} :\ \frac{f(z)}{z} \neq 0 \text{ and } \left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| < \lambda,\ z \in \mathbb{E} \right\},$$

where $0 < \lambda \leq 1$ and μ is a complex number and estimated the range of parameters λ and μ such that the functions in the class $U(\lambda,\ \mu)$ are starlike or spirallike. Note that $U(\lambda,\ \mu) = U(\lambda,\ \mu,\ 1 + \lambda z)$ for $0 < \lambda \leq 1$.

Obradovič et al. [3] proved:

Theorem 1.1. If $f \in \mathcal{A}$ satisfies the inequality

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < 1,\ z \in \mathbb{E},$$

then f is starlike.

Irmak and Şan [4] proved the following result.

Theorem 1.2. If $f \in \mathcal{A}$ satisfies the inequality

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \beta,\ z \in \mathbb{E},\ 0 < \beta \leq 1,$$

then f is strongly starlike function of order β.

In the present paper, we study the differential operator

$$\frac{zf''(z)}{f'(z)} + (\mu + 1)\left(1 - \frac{zf'(z)}{f(z)}\right)$$

and write the sufficient conditions in terms of this operator for $f \in \mathcal{A}$ to be a member of the class $U(\lambda,\ \mu,\ q(z))$ and consequently of $U(\lambda,\ \mu)$. In particular, we also derive some sufficient conditions in terms of the differential operator $1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}$ which extend the above results of Obradovič et al.[3] and Irmak and Şan [4] for $f \in \mathcal{A}$ to be starlike and strongly starlike.

To prove our main result, we shall use the following lemma of Miller and Mocanu [1] (page 76).

Lemma 1.1. Let $q,\ q(z) \neq 0$ be univalent in $\mathbb{E}$ such that $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$. If an analytic function $p,\ p(z) \neq 0$ in $\mathbb{E}$, satisfies the differential subordination

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)} = h(z),$$

then

$$p(z) \prec q(z) = \exp\left[\int_0^z \frac{h(t)}{t}\right]$$

and q is the best dominant.

§2. Main result and applications

Theorem 2.1. Let $q,\ q(z) \neq 0$ be univalent in $\mathbb{E}$ such that $\dfrac{zq'(z)}{q(z)}(= h(z))$ is starlike in $\mathbb{E}$. If $f \in \mathcal{A},\ f'(z)\left(\dfrac{z}{f(z)}\right)^{\mu+1} \neq 0$ for all z in $\mathbb{E}$, satisfies the differential subordination

$$\frac{zf''(z)}{f'(z)} + (\mu+1)\left(1 - \frac{zf'(z)}{f(z)}\right) \prec h(z),\ z \in \mathbb{E},$$

then

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec q(z) = \exp\left[\int_0^z \frac{h(t)}{t}\right],$$

i.e., $f \in U(\lambda,\ \mu,\ q(z))$ and q is the best dominant. Here μ is a complex number.

Proof. By setting $p(z) = f'(z)\left(\dfrac{z}{f(z)}\right)^{\mu+1}$ in Lemma 1.1, proof follows.

Remark 2.1. Consider the dominant $q(z) = \dfrac{1 + (1-2\alpha)z}{1-z},\ 0 \leq \alpha < 1,\ z \in \mathbb{E}$ in above theorem, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(1 + \frac{1}{1-z} - \frac{(1-2\alpha)z}{1+(1-2\alpha)z}\right) > 0,\ z \in \mathbb{E}$$

for all $0 \leq \alpha < 1$. Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and we immediately get the following result.

Theorem 2.2. For a complex number μ, if $f \in \mathcal{A},\ f'(z)\left(\dfrac{z}{f(z)}\right)^{\mu+1} \neq 0$ for all z in $\mathbb{E}$, satisfies the differential subordination

$$\frac{zf''(z)}{f'(z)} + (\mu+1)\left(1 - \frac{zf'(z)}{f(z)}\right) \prec \frac{2(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)},\ z \in \mathbb{E},$$

then

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec \frac{1+(1-2\alpha)z}{1-z},\ 0 \leq \alpha < 1,\ z \in \mathbb{E}.$$

Taking $\mu = 0$ in above theorem, we obtain the following result of Billing [5].

Corollary 2.1. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies the condition

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{2(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)},\ z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z},\ 0 \leq \alpha < 1,\ z \in \mathbb{E},$$

i.e., f is starlike of order α.

Remark 2.2. Setting $\alpha = 0$ in the above corollary, we obtain: Suppose $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{2z}{1-z^2} = F(z), \tag{2}$$

then f is starlike, where $F(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} :\ \Re(w) = 0,\ |\Im(w)| \geq 1\}$.

This result extends the region of variability of the differential operator $1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}$ largely over the result of Obradovič et al. [3] stated in Theorem 1.1. When the above said operator takes the values in the unit disk, then $f \in \mathcal{A}$ is starlike by Theorem 1.1. But in view of the result in (2), the same operator can take values in the complex plane $\mathbb{C}$ except two slits $\Re(w) = 0,\ |\Im(w)| \geq 1$. The dark portion of the Figure 2.1 is the region of variability of the differential operator $1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}$ in view of Theorem 1.1 whereas the region of variability is extended to the total shaded region (dark + light) in the light of the result in (2).

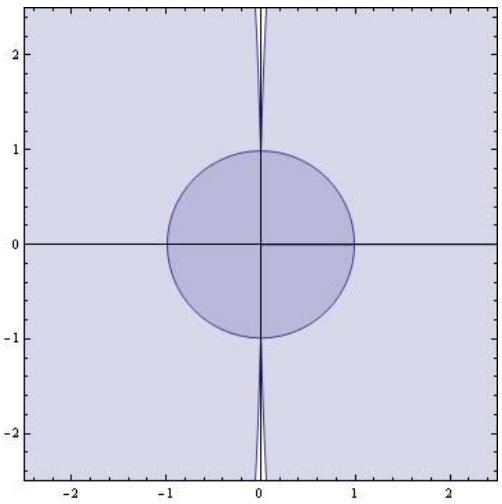

Figure 2.1

Remark 2.3. Consider the dominant $q(z) = 1 + \lambda z,\ 0 < \lambda \leq 1,\ z \in \mathbb{E}$ in Theorem 2.1, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1}{1+\lambda z}\right) > 0,\ z \in \mathbb{E}$$

for all $0 < \lambda \leq 1$. Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and we have the following result.

Theorem 2.3. Suppose $f \in \mathcal{A},\ f'(z)\left(\dfrac{z}{f(z)}\right)^{\mu+1} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$\frac{zf''(z)}{f'(z)} + (\mu+1)\left(1 - \frac{zf'(z)}{f(z)}\right) \prec \frac{\lambda z}{1+\lambda z},\ z \in \mathbb{E},$$

where μ is a complex number, then

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1\right| < \lambda,\ 0 < \lambda \leq 1,\ z \in \mathbb{E},$$

i.e., $f \in U(\lambda,\ \mu)$.

Selecting $\mu = 0$ in above theorem, we obtain:

Corollary 2.2. If $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies the condition

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{\lambda z}{1+\lambda z},\ z \in \mathbb{E},$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda,\ 0 < \lambda \leq 1,\ z \in \mathbb{E}.$$

Remark 2.4. Let the dominant in Theorem 2.1 be $q(z) = \dfrac{\alpha(1-z)}{\alpha - z},\ \alpha > 1,\ z \in \mathbb{E}$. A little calculation yields

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1}{1-z} + \frac{z}{\alpha - z}\right) > 0,\ z \in \mathbb{E}.$$

Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and we obtain the following result.

Theorem 2.4. Let $\alpha > 1$ be a real number and let $f \in \mathcal{A},\ f'(z)\left(\dfrac{z}{f(z)}\right)^{\mu+1} \neq 0$ for all z in $\mathbb{E}$, satisfy the differential subordination

$$\frac{zf''(z)}{f'(z)} + (\mu+1)\left(1 - \frac{zf'(z)}{f(z)}\right) \prec \frac{(1-\alpha)z}{(1-z)(\alpha - z)},\ z \in \mathbb{E},$$

then

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec \frac{\alpha(1-z)}{\alpha - z},\ z \in \mathbb{E},$$

where μ is a complex number.

Selecting $\mu = 0$ in above theorem, we obtain:

Corollary 2.3. If $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{(1-z)(\alpha - z)},\ z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha(1-z)}{\alpha - z},\ \alpha > 1,\ z \in \mathbb{E}.$$

Remark 2.5. Selecting the dominant $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\delta},\ 0 < \delta \leq 1,\ z \in \mathbb{E}$ in Theorem 2.1, we have

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1+z^2}{1-z^2}\right) > 0,\ z \in \mathbb{E}.$$

Therefore, $\dfrac{zq'(z)}{q(z)}$ is starlike in $\mathbb{E}$ and we obtain the following result.

Theorem 2.5. Let $f \in \mathcal{A},\ f'(z)\left(\dfrac{z}{f(z)}\right)^{\mu+1} \neq 0$ for all z in $\mathbb{E}$, satisfy the differential subordination

$$\frac{zf''(z)}{f'(z)} + (\mu+1)\left(1 - \frac{zf'(z)}{f(z)}\right) \prec \frac{2\delta z}{1-z^2},\ z \in \mathbb{E},$$

then

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ 0 < \delta \leq 1,\ z \in \mathbb{E}.$$

Taking $\mu = 0$ in above theorem, we get:

Corollary 2.4. If $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{2\delta z}{1-z^2},\ z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ 0 < \delta \leq 1,\ z \in \mathbb{E},$$

i.e., f is strongly starlike of order δ.

Remark 2.6. We, here, make the comparison of the result in above corollary with the result in Theorem 1.2. We notice that the region of variability of the differential operator $1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}$ is extended in Corollary 2.4 over the result in Theorem 1.2. To justify our claim, we consider the following particular case of both the results. For $\delta = 1/2$ in Corollary 2.4, we obtain: Suppose $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0$ for all z in $\mathbb{E}$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{z}{1-z^2} = G(z), \tag{3}$$

then f is strongly starlike of order $1/2$ where $G(\mathbb{E}) = \mathbb{C} \setminus \{w :\ \Re(w) = 0,\ |\Im(w)| \geq 1/2\}$.

For $\beta = 1/2$ in Theorem 1.2, we get: If $f \in \mathcal{A}$ satisfies the inequality

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{1}{2},\ z \in \mathbb{E}, \tag{4}$$

then f is strongly starlike of order $1/2$.

We see that in view of (4) the differential operator $1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}$ takes values in the disk of radius $1/2$ with center at origin whereas by (3), the same operator can take values in the complex plane $\mathbb{C}$ except two slits $\Re(w)=0,\ |\Im(w)|\geq 1/2$ for the same conclusion i.e. f is strongly starlike of order $1/2$. We also justify our claim pictorially in Figure 2.2. The dark shaded portion of the figure shows the region of variability of the above said operator in view of the result in (4) and in the light of the result in (3), the region of variability is the total shaded region (dark + light).

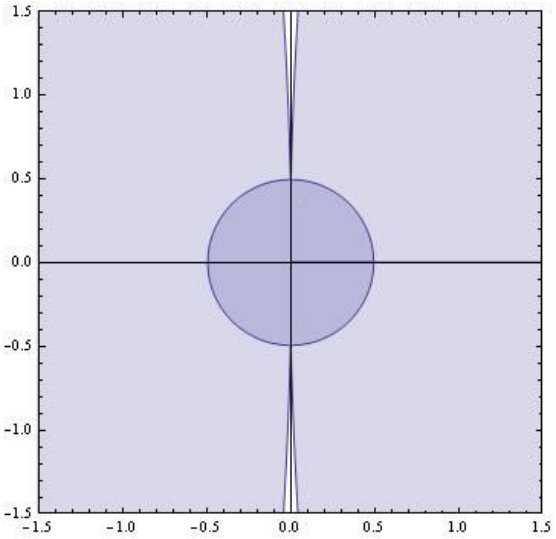

Figure 2.2

References

[1] S. S. Miller and P. T. Mocanu, Differential Subordinations, Theory and Applications, Marcel Dekker, New York and Basel, 2000, No. 225.

[2] R. Fournier and S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, Complex Variables and Elliptic Equations, **52**(2007), No. 1, 1-8.

[3] M. Obradovič, S. Ponnusamy, V. Singh and P. Vasundhra, A class of locally univalent functions defined by a differential inequality, Complex Variables and Elliptic Equations, **52**(2007), No. 1, 1-8.

[4] H. Irmak and M. Şan, Some relations betwwen certain inequalities concerning analytic and univalent functions, Applied Mathematics Letters, **23**(2010), 897-901.

[5] S. S. Billing, Differential sandwich-type results and criteria for starlikeness, Rend. Sem. Mat. Univ. Politec, Torino, **69**(2011), No. 1, 57-71.

Scientia Magna
Vol. 7 (2011), No. 4, 42-48

On ideal convergence in topological groups

Bipan Hazarika

Department of Mathematics, Rajiv Gandhi University Rono Hills,
Doimukh, 791112, Arunachal Pradesh, India
E-mail: bh_rgu@yahoo.co.in

Abstract An ideal I is a family of subsets of positive integers $\mathbf{N}$ which is closed under taking finite unions and subsets of its elements. In [9], Kostyrko et. al introduced the concept of ideal convergence as a sequence (x_k) of real numbers is said to be I-convergent to a real number x_0, if for each $\varepsilon > 0$ the set $\{k \in \mathbf{N} : |x_k - x_0| \geq \varepsilon\}$ belongs to I. In this article we introduced I-convergence of sequences in topological groups and extensions of a decomposition theorem and a completeness theorem to the topological group setting are proved.

Keywords Ideal, I-convergence, I-cauchy, topological group.
2010 Mathematics Subject Classifications: 40A05, 40G15, 40J05, 22A05.

§1. Introduction

The concept of ideal convergence as a generalization of statistical convergence, and any concept involving statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by Fast [7], and Schoenberg [13], independently for the real sequences. Later on it was further investigated from sequence point of view and linked with the summability theory by Fridy [8] and many others. The idea is based on the notion of natural density of subsets of $\mathbf{N}$, the set of positive integers, which is defined as follows: The natural density of a subset of $\mathbf{N}$ is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{n\to\infty} |\{k \in E : \ k \leq n\}|,$$

where the vertical bar denotes the cardinality of the respective set. In [5], Çakalli introduced this notion in topological Hausdroff groups.

The notion of I-convergence was initially introduced by Kostyrko, et. al [9] as a generalization of statistical convergence which is based on the structure of the ideal I of subset of natural numbers $\mathbf{N}$. Later on it was further investigated from sequence space point of view and linked with summability theory by Šalát, et. al [11−12], Tripathy and Hazarika [14−17] and many other authors.

A non-empty family of sets $I \subseteq P(\mathbf{N})$ (power set of $\mathbf{N}$) is called an ideal in $\mathbf{N}$ if and only if

(i) for each $A,\ B \in I$, we have $A \cup B \in I$;

(ii) for each $A \in I$ and $B \subset A$, we have $B \in I$.

A family $F \subseteq P(\mathbf{N})$ is called a filter on $\mathbf{N}$ if and only if

(i) $\phi \notin F$;

(ii) for each $A,\ B \in F$, we have $A \cap B \in F$;

(iii) for each $A \in F$ and $B \supset A$, we have $B \in F$.

An ideal I is called non-trivial if $I \neq \phi$ and $\mathbf{N} \notin I$. It is clear that $I \subseteq P(\mathbf{N})$ is a non-trivial ideal if and only if the class $F = F(I) = \mathbf{N} - A :\ A \in I$ is a filter on $\mathbf{N}$. The filter $F(I)$ is called the filter associated with the ideal I. A non-trivial ideal $I \subseteq P(\mathbf{N})$ is called an admissible ideal in $\mathbf{N}$ if it contains all singletons, i.e., if it contains $\{\{x\} :\ x \in \mathbf{N}\}$.

Recall that a sequence $x = (x_k)$ of points in $\mathbf{R}$ is said to be I-convergent to a real number x_0 if $\{k \in \mathbf{N} :\ |x_k - x_0| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ (see [9]). In this case we write $I - \lim x_k = x_0$.

Let (x_k) and (y_k) be two real sequences, then we say that $x_k = y_k$ for almost all k related to I (*a. a. k. r. I*) if the set $\{k \in \mathbf{N} :\ x_k \neq y_k\}$ belongs to I.

Throughout the article we consider I to be a non-trivial admissible ideal of $\mathbf{N}$.

The purpose of this article is to give certain characterizations of I-convergent sequences in topological groups and to obtain extensions of a decomposition theorem and a completeness theorem to topological groups.

§2. I-convergence in topological groups

Throughout the article X will denotes the topological Hausdroff group, written additively, which satisfies the first axiom of countability.

Definition 2.1. A sequence (x_k) of points in X is said to be I-convergent to an element x_0 of X if for each neighbourhood V of 0 such that the set

$$\{k \in \mathbf{N} :\ x_k - x_0 \notin V\} \in I$$

and it is denoted by $I - \lim_{k \to \infty} x_k = x_0$.

Definition 2.2. A sequence (x_k) of points in X is said to be I-Cauchy in X if for each neighbourhood V of 0, there is an integer $n(V)$ such that the set

$$\{k \in \mathbf{N} :\ x_k - x_{n(V)} \notin V\} \in I.$$

Throughout the article $s(X)$, $c^I(X)$, $c_0^I(X)$ and $C^I(X)$ denote the set of all X-valued sequences, X-valued I-convergent sequences, X-valued I-null sequences and X-valued I-Cauchy sequences in X, respectively.

Theorem 2.1. A sequence (x_k) is I-convergent if and only if for each neighbourhood V of 0 there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $\lim_{r \to \infty} x_{k'(r)} = x_0$ and

$$\{k \in \mathbf{N} :\ x_k - x_{k'(r)} \notin V\} \in I.$$

Proof. Let $x = (x_k)$ be a sequence in X such that $I - \lim_{k\to\infty} x_k = x_0$. Let $\{V_n\}$ be a sequence of nested base of neighbourhoods of 0. We write $E^{(i)} = \{k \in \mathbf{N} :\ x_k - x_0 \notin V_i\}$ for any positive integer i. Then for each i, we have $E^{(i+1)} \subset E^{(i)}$ and $E^{(i)} \in F(I)$. Choose $n(1)$ such that $k > n(1)$, then $E^{(i)} \neq \phi$. Then for each positive integer r such that $n(p+1) \leq r < n(2)$, choose $k'(r) \in E^{(p)}$ i.e. $x_{k'(r)} - x_0 \in V_1$, In general, choose $n(p+1) > n(p)$ such that $r > n(p+1)$, then $E^{(p+1)} \neq \phi$. Then for all r satisfying $n(p) \leq r < n(p+1)$, choose $k'(r) \in E^{(p)}$ i.e. $x_{k'(r)} - x_0 \in V_p$. Also for every neighbourhood V of 0, there is a symmetric neighbourhood W of 0 such that $W \cup W \subset V$. Then we get

$$\{k \in \mathbf{N} :\ x_k - x_{k'(r)} \notin V\} \subset \{k \in \mathbf{N} :\ x_k - x_0 \notin W\} \cup \{r \in \mathbf{N} :\ x_{k'(r)} - x_0 \notin W\}.$$

Since $I - \lim_{k\to\infty} x_k = x_0$, therefore there is a neighbourhood W of 0 such that

$$\{k \in \mathbf{N} :\ x_k - x_0 \notin W\} \in I$$

and $\lim_{r\to\infty} x_{k'(r)} = x_0$ implies that

$$\{r \in \mathbf{N} :\ x_{k'(r)} - x_0 \notin W\} \in I.$$

Thus we have

$$\{k \in \mathbf{N} :\ x_k - x_{k'(r)} \notin V\} \in I.$$

Next suppose for each neighbourhood V of 0 there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $\lim_{r\to\infty} x_{k'(r)} = x_0$ and

$$\{k \in \mathbf{N} :\ x_k - x_{k'(r)} \notin V\} \in I.$$

Since V is a neighbourhood of 0, we may choose a symmetric neighbourhood W of 0 such that $W \cup W \subset V$. Then we have

$$\{k \in \mathbf{N} :\ x_k - x_0 \notin V\} \subset \{k \in \mathbf{N} :\ x_k - x_{k'(r)} \notin W\} \cup \{r \in \mathbf{N} :\ x_{k'(r)} - x_0 \notin W\}.$$

Since both the sets on the right hand side of the above relation belong to I. Therefore

$$\{k \in \mathbf{N} :\ x_k - x_0 \notin V\} \in I.$$

This completes the proof of the theorem.

Theorem 2.2. If $\lim_{k\to\infty} x_k = x_0$ and $I - \lim_{k\to\infty} y_k = 0$, then $I - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k$.

Proof. Let V be any neighbourhood of 0. Then we may choose a symmetric neighbourhood W of 0 such that $W \cup W \subset V$. Since $\lim_{k\to\infty} x_k = x_0$, then there exists an integer n_0 such that $k \geq n_0$ implies that $x_k - x_0 \in W$. Hence

$$\{k \in \mathbf{N} :\ x_k - x_0 \notin W\} \in I.$$

By assumption $I - \lim_{k\to\infty} y_k = 0$, then we have $\{k \in \mathbf{N} :\ y_k \notin W\} \in I$. Thus

$$\{k \in \mathbf{N} :\ (x_k - x_0) + y_k \notin V\} \subset \{k \in \mathbf{N} :\ x_k - x_0 \notin W\} \cup \{k \in \mathbf{N} :\ y_k \notin W\} \in I.$$

This implies that $I - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k$.

Theorem 2.3. If a sequence (x_k) is I-convergent to x_0, then there are sequences (y_k) and (z_k) such that $I - \lim_{k\to\infty} y_k = x_0$ and $x_k = y_k + z_k$, for all $k \in \mathbf{N}$ and $\{k \in \mathbf{N} : \ x_k \neq y_k\} \in I$ and (z_k) is a I-null sequence.

Proof. Let $\{V_i\}$ be a nested base of neighbourhood of 0. Take $n_0 = 0$ and choose an increasing sequence (n_i) of positive integer such that

$$\{k \in \mathbf{N} : \ x_k - x_0 \notin V_i\} \in I \text{ for } k > n_0.$$

Let us define the sequences (y_k) and (z_k) as follows:

$$y_k = x_k \quad \text{and} \quad z_k = 0, \quad \text{if} \quad 0 < k \leq n_1$$

and suppose $n_i < n_{i+1}$, for $i \geq 1$,

$$y_k = x_k \quad \text{and} \quad z_k = 0, \quad \text{if} \quad x_k - x_0 \in V_i,$$

$$y_k = x_0 \quad \text{and} \quad z_k = x_k - x_0, \quad \text{if} \quad x_k - x_0 \notin V_i.$$

We have to show that (i) $I - \lim_{k\to\infty} y_k = x_0$, (ii) (z_k) is a I-null sequence.

(i) Let V be any neighbourhood of 0. We may choose a positive integer i such that $V_i \subset V$. Then $y_k - x_0 = x_k - x_0 \in V_i$, for $k > n_i$.

If $x_k - x_0 \notin V_i$, then $y_k - x_0 = x_0 - x_0 = 0 \in V$. Hence $I - \lim_{k\to\infty} y_k = x_0$.

(ii) It is enough to show that $\{k \in \mathbf{N} : \ z_k \neq 0\} \in I$. For any neighbourhood V of 0, we have

$$\{k \in \mathbf{N} : \ z_k \notin V\} \subset \{k \in \mathbf{N} : \ z_k \neq 0\}.$$

If $n_p < k \leq n_{p+1}$, then

$$\{k \in \mathbf{N} : \ z_k \neq 0\} \subset \{k \in \mathbf{N} : \ x_k - x_0 \notin V_p\}.$$

If $p > i$ and $n_p < k \leq n_{p+1}$, then

$$\{k \in \mathbf{N} : \ z_k \neq 0\} \subset \{k \in \mathbf{N} : \ x_k - x_0 \notin V_p\} \subset \{k \in \mathbf{N} : \ x_k - x_0 \notin V_i\}.$$

This implies that $\{k \in \mathbf{N} : \ z_k \neq 0\} \in I$. Hence (z_k) is a I-null sequence.

§3. I-sequential compactness

Definition 3.1. Let $A \subset X$ and $x_0 \in X$. Then x_0 is in the I-sequential closure of A if there is a sequence (x_k) of points in A such that $I - \lim_{k\to\infty} x_k = x_0$.

We denote I-sequential closure of a set A by $\overline{A}^I$. We say that a set is I-sequentially closed if it contains all of the points in its I-sequential closure.

By a method of sequential convergence, we mean an additive function B defined on a subgroup of $s(X)$, denoted by $c^B(X)$ into X.

Definition 3.2. A sequence $x = (x_k)$ is said to be B-convergent to x_0 if $x \in c^B(X)$ and $B(x) = x_0$.

Definition 3.3. A method B is called regular if every convergent sequence $x = (x_k)$ is B-convergent with $B(x) = \lim x$.

From Definition 3.3, it is clear that, if $B = I$, then I is a regular sequential method.

Definition 3.4. A point x_0 is called a I-sequential accumulation point of A (or is in the I-sequential derived set) if there is a sequence $x = (x_k)$ of points in $A - \{x_0\}$ such that $I\text{-}\lim x = x_0$.

Definition 3.5. A subset A of X is called I-sequentially countably compact if any infinite subset A has at least one I-sequentially accumulation point in A.

Definition 3.6. A subset A of X is called I-sequentially compact if $x = (x_k)$ is a sequence of points of A, there is a subsequence $y = (y_{k_n})$ of x with $I\text{-}\lim y = x_0 \in A$.

Definition 3.7. A function f is called I-sequentially continuous at $x_0 \in X$ provided that whenever a sequence $x = (x_k)$ of points of X is I-convergent to x_0 then the sequence $f(x) = f((x_k))$ is I-convergent to $f(x_0)$. For a subset A of X, f is I-sequentially continuous on A if it is I-sequentially continuous at every $x_0 \in A$ and is I-sequentially continuous if it is I-sequentially continuous on X.

Lemma 3.1. Any I-convergent sequence of points in X with a I-limit x_0 has a convergent subsequence with the same limit x_0 in the ordinary sense.

The proof of the lemma follows from [11] and [17].

From the Lemma 3.1, it is clear that I is a regular subsequential method.

Theorem 3.2. Any I-sequentially closed subset of a I-sequentially compact subset of X is I-sequentially compact.

Proof. Let A be a I-sequentially compact subset of X and E be a I-sequentially closed subset of A. Let $x = (x_k)$ be a sequence of points in E. Then x is a sequence of points in A. Since A is I-sequentially compact, there exists a subsequence $y = (y_{k_n})$ of the sequence (x_k) such that $I - \lim\ y \in A$. The subsequence (y_{k_n}) is also a sequence of points in E and E is I-sequentially closed, therefore $I - \lim\ y \in E$. Thus $x = (x_k)$ has a I-convergent subsequence with $I - \lim y \in E$, so E is I-sequentially compact.

Theorem 3.3. It Any I-sequentially compact subset of X is I-sequentially closed.

Proof. Let A be any I-sequentially compact subset of X. For any $x_0 \in \overline{A}^I$, then there exists a sequence $x = (x_k)$ be a sequence of points in A such that $I - \lim x = x_0$. Since I is a subsequential method, there is a subsequence $y = (y_r) = (x_{k_r})$ of the sequence $x = (x_k)$ such that $\lim_{r\to\infty} x_{k_r} = x_0$. Since I is regular, so $I - \lim y = x_0$. By I-sequentially compactness of A is, there is a subsequence $z = (z_r)$ of $y = (y_r)$ such that $I - \lim z = y_0 \in A$. Since $\lim_{r\to\infty} z_r = x_0$ and I is regular, so $I - \lim z = x_0$. Thus $x_0 = y_0$ and hence $x_0 \in A$. Hence A is I-sequentially closed.

Corollary 3.4. Any I-sequentially compact subset of X is closed in the ordinary sense.

Theorem 3.5. A subset of X is I-sequentially compact if and only if it is I-sequentially countably compact.

Proof. Let A be any I-sequentially compact subset of X and E be an infinite subset of A. Let $x = (x_k)$ be a sequence of different points of E. Since A is I-sequentially compact, this implies that the sequence x has a convergent subsequence $y = (y_r) = (x_{k_r})$ with $I - \lim y = x_0$. Since I is subsequential method, y has a convergent subsequence $z = (z_r)$ of the subsequence y

with $\lim_{r\to\infty} z_r = x_0$. By the regularity of I, we obtain that x_0 is a I-sequentially accumulation point of E. Then A is I-sequentially countably compact.

Next suppose A is any I-sequentially countably compact subset of X. Let $x = (x_k)$ be a sequence of different points in A. Put $G = \{x_k : k \in \mathbf{N}\}$. If G is finite, then there is nothing to prove. If G is infinite, then G has a I-sequentially accumulation point in A. Also each set $G_n = \{x_n : n \geq k\}$, for each positive integer n, has a I-sequentially accumulation point in A. Therefore $\cap_{n=1}^{\infty} \overline{G_n}^I \neq \phi$. So there is an element $x_0 \in A$ such that $x_0 \in \overline{G_n}^I$. Since I is a regular subsequential method, so $x_0 \in \overline{G_n}$. Then there exists a subsequence $z = (z_r) = (x_{k_r})$ of the sequence $x = (x_k)$ with $I - \lim z \in A$. This completes the proof.

Corollary 3.6. A subset of X is I-sequentially compact if and only if it is sequentially countably compact in the ordinary sense.

Corollary 3.7. A subset of X is I-sequentially compact if and only if it is countably compact in the ordinary sense.

Theorem 3.8. The I-sequential continuous image of any I-sequentially compact subset of X is I-sequentially compact.

Proof. Let f be any I-sequentially continuous function on X and A be any I-sequentially compact subset of X. Let $y = (y_k) = (f(x_k))$ be a sequence of points in $f(A)$. Since A is I-sequentially compact, there exists a subsequence $z = (z_r) = (x_{k_r})$ of the sequence $x = (x_k)$ with $I - \lim zA$. Then $f(z) = (f(z_r)) = (f(x_{k_r}))$ is a subsequence of the sequence y. Since f is I-sequentially continuous, $I - \lim(f(z_r)) = f(x) \in f(A)$. Then $f(A)$ is I-sequentially compact.

Corollary 3.9. A I-sequentially continuous image of any sequentially compact subset of X is sequentially compact.

Theorem 3.10. If X is I-sequentially compact, then X is complete.

Proof. Let $x = (x_k)$ be a Cauchy sequence of points of X. As X is I-sequentially compact, there exists a convergent subsequence $y = (y_{k_r})$ of the sequence x such that $I - \lim y = x_0 \in A$. By the Theorem 2.1, there exists a subsequence $z = (z_{k_r})$ of the sequence y such that $\lim_{r\to\infty} z_{k_r} = x_0$. Hence (x_k) is converges. This completes proof of the theorem.

References

[1] J. Antoni and T. Šalát, On the A-continuity of real functions, Acta Math. Univ. Comenian, **39**(1980), 159-164.

[2] V. Balaz, J. Cervenansky, P. Kostrysko and T. Šalát, I-convergence and I-continuity of real functions, Acta Math., **5**(2002), 43-50.

[3] J. Boos, Classical and Modern Methods in Summability, Oxford Univ. Press, 2000.

[4] H. Çakalli and B. Thorpe, On summability in topological groups and a theorem of D. L. Prullage, Ann. Soc. Math. Polon. Comm. Math. Ser I, **29**(1990), 139-148.

[5] H. Çakalli, On statistical convergence in topological groups, Pure Appl. Math. Sci., **43**(1996), No. 1-2, 27-31.

[6] J. Connor and K. G. Grosse-Erdmann, Sequential definition of continuity for real functions, Rocky Mountains Jour., **33**(2003), No. 1, 93-121.

[7] H. Fast, Sur la convergence statistique, Colloq. Math., **2**(1951), 241-244.

[8] J. A. Fridy, On statistical convergence, Analysis, **5**(1985), 301-313.

[9] P. Kostrysko, T. Šalát and W. Wilczyński, I-convergence, Real Anal. Exchange, **26**(2000-2001), No. 2, 669-686.

[10] D. L. Prullage, Summability in topological group, Math. Z., **96**(1967), 259-279.

[11] T. Šalát, B. C. Tripathy and M. Ziman, On some properties of I-convergence, Tatra Mt. Math. Publ., **28**(2004), 279-286.

[12] T. Šalát, B. C. Tripathy and M. Ziman, On I-convergence field, Italian Jour. Pure Appl. Math., **17**(2005), 45-54.

[13] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, **66**(1951), 361-375.

[14] B. C. Tripathy and B. Hazarika, I-convergent sequence spaces associated with multiplier sequences, Math. Ineq. Appl., **11**(2008), No. 3, 543-548.

[15] B. C. Tripathy and B. Hazarika, Paranorm I-convergent sequence spaces, Math. Slovaca, **59**(2009), No. 4, 485-494.

[16] B. C. Tripathy and B. Hazarika, Some I-convergent sequence spaces defined by Orlicz function, Acta Mathematica Applicatae Sinica, **27**(2011), No. 1, 149-154.

[17] B. C. Tripathy and B. Hazarika, I-Monotonic and I-Convergent Sequences, Kyungpook Math. J., **51**(2011), No. 2, 233-239.

Scientia Magna
Vol. 7 (2011), No. 4, 49-81

A summation formula including recurrence relation

Salahuddin

P. D. M College of Engineering, Bahadurgarh, Haryana, India
E-mail: sludn@yahoo.com vsludn@gmail.com

Abstract The aim of the present paper is to establish a summation formula based on half argument using Gauss second summation theorem.

Keywords Contiguous relation, gauss second summation theorem, recurrence relation.
A. M. S. Subject Classification 2000: 33C60, 33C70.

§1. Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ccc} a_1,\ a_2,\ \cdots,\ a_A & ; & \\ & & z \\ b_1,\ b_2,\ \cdots,\ b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}, $$

or

$$ {}_AF_B\left[\begin{array}{ccc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ccc} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!}. \tag{1} $$

where the parameters $b_1,\ b_2,\ \cdots,\ b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation is defined by

[Andrews p.**363**(9.16), E.D.p.**51**(10), H.T.F.I p.**103**(32)]

$$ (a-b)\ {}_2F_1\left[\begin{array}{ccc} a,\ b\ ; & & \\ & & z \\ c\ ; & & \end{array}\right] = a\ {}_2F_1\left[\begin{array}{ccc} a+1,\ b\ ; & & \\ & & z \\ c\ ; & & \end{array}\right] - b\ {}_2F_1\left[\begin{array}{ccc} a,\ b+1\ ; & & \\ & & z \\ c\ ; & & \end{array}\right]. \tag{2} $$

Gauss second summation theorem is defined by [Prud., **491**(7.3.7.5)]

$$ \begin{aligned} {}_2F_1\left[\begin{array}{ccc} a,\ b\ ; & & \\ & & \frac{1}{2} \\ \frac{a+b+1}{2}\ ; & & \end{array}\right] &= \frac{\Gamma(\frac{a+b+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \\ &= \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(b)\ \Gamma(\frac{a+1}{2})}. \end{aligned} \tag{3} $$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prud., p.**491**(7.3.7.3)]

$$ {}_2F_1\left[\begin{array}{ccc} a,\ b & ; & \\ \frac{a+b-1}{2} & ; & \frac{1}{2}\end{array}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} + \frac{2\,\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\,\Gamma(b)}\right]. \tag{4} $$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$$ {}_2F_1\left[\begin{array}{ccc} a,\ b & ; & \\ \frac{a+b-1}{2} & ; & \frac{1}{2}\end{array}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right]. \tag{5} $$

Recurrence relation is defined by

$$ \Gamma(z+1) = z\,\Gamma(z). \tag{6} $$

§2. Main summation formula

$$
\begin{aligned}
&{}_2F_1\left[\begin{array}{ccc} a,\ \ b & ; & \\ \frac{a+b+32}{2} & ; & \frac{1}{2}\end{array}\right] \\
&= \frac{2^b\,\Gamma(\frac{a+b+32}{2})}{(a-b)\,\Gamma(b)} \\
&\quad\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{32768(1428329123020800a - 2322150583173120a^2 + 1606274243887104a^3)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}\right.\right. \\
&\quad+\frac{32768(-636906005299200a^4 + 163554924216320a^5)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)} \\
&\quad+\frac{32768(-29011643781120a^6 + 3688786669568a^7)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)} \\
&\quad+\frac{32768(-343226083200a^8 + 23578343360a^9)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)} \\
&\quad\left.\left.+\frac{32768(-1193992800a^{10} + 43995952a^{11} - 1146600a^{12})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}\right.\right.
\end{aligned}
$$

$$+\frac{32768(20020a^{13}-210a^{14}+a^{15}+1428329123020800b+8525223163330560a^2b)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-2182234506854400a^3b+1969113895649280a^4b)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-264966591528960a^5b+82650036080640a^6b)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-6233334240000a^7b+920583637440a^8b)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-39161191680a^9b+3018684240a^{10}b-67407600a^{11}b)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(2691780a^{12}b-24360a^{13}b+435a^{14}b)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(2322150583173120b^2+8525223163330560ab^2)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(5852172660080640a^3b^2-528194276167680a^4b^2+497028396625920a^5b^2)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-30401221536000a^6b^2+9824497355520a^7b^2)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-3635128238840a^8b^2+54807626640a^9b^2)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-1104760800a^{10}b^2+85096440a^{11}b^2-712530a^{12}b^2)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(27405a^{13}b^2+1606274243887104b^3)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(2182234506854400ab^3+5852172660080640a^2b^3+1153266031104000a^4b^3)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-46525375584000a^5b^3+42437501433600a^6b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-1285763673600a^7b^3+404597084400a^8b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(-7227090000a^9b^3+1052508600a^{10}b^3-8143200a^{11}b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(593775a^{12}b^3+6369060052992200b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(1969113895649280ab^4+528194276167680a^2b^4+1153266031104000a^3b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(858001202208800a^5b^4-1679965963200a^6b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(14383620216000a^7b^4-209804850000a^8b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(62301135000a^9b^4-429214500a^{10}b^4+58529250a^{11}b^4+1635549242163200b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(264966591528960ab^5+497028396625920a^2b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(46525375584000a^3b^5+858001202208800a^4b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(2667136741920a^6b^5-24708348000a^7b^5+19527158700a^8b^5-109254600a^9b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(300450150a^{10}b^5+290116437811200b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(82650360806400ab^6+304012215360000a^2b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(424375014336000a^3b^6+16799659632000a^4b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(26671367419200a^5b^6+341714772000a^7b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(-1197598500a^8b^6+864932250a^9b^6+36887866695680b^7+62333342400000ab^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(98244973555200a^2b^7+12857636736000a^3b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(14383620216000a^4b^7+247083480000a^5b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(341714772000a^6b^7+1454226750a^8b^7+3432260832000b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(9205836374400ab^8+3635128238400a^2b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(4045970844000a^3b^8+209804850000a^4b^8+195271587000a^5b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(1197598500a^6b^8+1454226750a^7b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(235783433600b^9+391611916800ab^9+548076266400a^2b^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(72270900000a^3b^9+62301135000a^4b^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(1092546000a^5b^9+864932250a^6b^9+11939928000b^{10})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{32768(30186842400ab^{10}+11047608000a^2b^{10})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(1052508600a^3b^{10}+42921450a^4b^{10}+30045015a^5b^{10})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(43995952b^{11}+67407600ab^{11})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(85096440a^2b^{11}+8143200a^3b^{11}+5852925a^4b^{11})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(1146600b^{12}+2691780ab^{12}+712530a^2b^{12})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(593775a^3b^{12}+20020b^{13}+24360ab^{13}+27405a^2b^{13}+210b^{14}+435ab^{14}+b^{15})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536b(1428329123020800+470423898685440a)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(2328961922629632a^2+381117697130496a^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(316734590500864a^4+29456251432960a^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(9279610167296a^6+511595950208a^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(77189562432a^8+2483214240a^9+193882832a^{10})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(3327896a^{11}+131404a^{12}+910a^{13})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(77189562432a^8+2483214240a^9+193882832a^{10})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(3327896a^{11}+131404a^{12}+910a^{13})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(15a^{14}-470423898685440b+5473573561958400ab+503214007025664a^2b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(1969528317345792a^3b+150112771248128a^4b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(112539365949440a^5b+5581588819072a^6b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(1638717738496a^7b+49573632864a^8b+6999713760a^9b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(116414584a^{10}b+8390512a^{11}b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(58058a^{12}b+2030a^{13}b+2328961922629632b^2-503214007025664ab^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(3412315664252928a^2b^2+129681106132992a^3b^2+423536993230848a^4b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(16628085832320a^5b^2+11189308499712a^6b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(300862325376a^7b^2+80994977424a^8b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(1263236520a^9b^2+163524504a^{10}b^2+1099332a^{11}b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(71253a^{12}b^2-381117697130496b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(1969528317345792ab^3-129681106132992a^2b^3+645921311784960a^3b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(11831795932800a^4b^3+33081548382720a^5b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(686621093760a^6b^3+413821722240a^7b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(5675445000a^8b^3+1386606000a^9b^3+8708700a^{10}b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(1017900a^{11}b^3+316734590500864b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-150112771248128ab^4+423536993230848a^2b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-11831795932800a^3b^4+47032752624000a^4b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(436682232000a^5b^4+1060441956000a^6b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(11019106800a^7b^4+5939968500a^8b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(32516250a^9b^4+7153575a^{10}b^4-294562514329 60b^5+112539365949440ab^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-16628085832320a^2b^5+33081548382720a^3b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-436682232000a^4b^5+1443061650240a^5b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(6518297520a^6b^5+13870291680a^7b^5+56728350a^8b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(27313650a^9b^5+9279610167296b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-5581588819072ab^6+11189308499712a^2b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-686621093760a^3b^6+1060441956000a^4b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-6518297520a^5b^6+18331241040a^6b^6+31935960a^7b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(59879925a^8b^6-5115959502 08b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(1638717738496ab^7-300862325376a^2b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(413821722240a^3b^7-11019106800a^4b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(13870291680a^5b^7-31935960a^6b^7+77558760a^7b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(77189562432b^8-49573632864ab^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(80994977424a^2b^8-5675445000a^3b^8+5939968500a^4b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-56728350a^5b^8+59879925a^6b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-2483214240b^9+6999713760ab^9-1263236520a^2b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(1386606000a^3b^9-32516250a^4b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(27313650a^5b^9+193882832b^{10}-116414584ab^{10})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(163524504a^2b^{10}-8708700a^3b^{10})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(7153575a^4b^{10}-3327896b^{11}+8390512ab^{11})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-1099332a^2b^{11}+1017900a^3b^{11}+131404b^{12})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536b(-58058ab^{12}+71253a^2b^{12}-910b^{13}+2030ab^{13}+15b^{14})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}\Bigg\}$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Bigg\{\frac{65536a(1428329123020800-470423898685440a+2328961922629632a^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(-381117697130496a^3+316734590500864a^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(-29456251432960a^5+9279610167296a^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(-511595950208a^7+77189562432a^8-2483214240a^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(193882832a^{10}-3327896a^{11})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(131404a^{12}-910a^{13}+15a^{14}+470423898685440b+5473573561958400ab)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(-503214007025664a^2b+1969528317345792a^3b-150112771248128a^4b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(112539365949440a^5b-5581588819072a^6b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(1638717738496a^7b-49573632864a^8b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(6999713760a^9b-116414584a^{10}b+8390512a^{11}b-58058a^{12}b+2030a^{13}b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(2328961922629632b^2+503214007025664ab^2+3412315664252928a^2b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(-129681106132992a^3b^2+423536993230848a^4b^2-16628085832320a^5b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(11189308499712a^6b^2-300862325376a^7b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(80994977424a^8b^2-1263236520a^9b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(163524504a^{10}b^2-1099332a^{11}b^2+71253a^{12}b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(381117697130496b^3+1969528317345792ab^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(129681106132992a^2b^3+645921311784960a^3b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(-11831795932800a^4b^3+33081548382720a^5b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(-686621093760a^6b^3+413821722240a^7b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(-5675445000a^8b^3+1386606000a^9b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(1017900a^{11}b^3-8708700a^{10}b^3+316734590500864b^4+150112771248128ab^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(423536993230848a^2b^4+11831795932800a^3b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(47032752624000a^4b^4-436682232000a^5b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(1060441956000a^6b^4-11019106800a^7b^4+5939968500a^8b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536a(-32516250a^9b^4+7153575a^{10}b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(294562514329 60b^5+112539365949440ab^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(166280858323 20a^2b^5+33081548382720a^3b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(436682232000a^4b^5+1443061650240a^5b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(-6518297520a^6b^5+13870291680a^7b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(-56728350a^8b^5+27313650a^9b^5+9279610167296b^6+5581588819072ab^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(111893084997 12a^2b^6+686621093760a^3b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(1060441956000a^4b^6+6518297520a^5b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(183312410 40a^6b^6-31935960a^7b^6+59879925a^8b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(5115959502 08b^7+1638717738496ab^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(300862325376a^2b^7+413821722240a^3b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(110191068 00a^4b^7+13870291680a^5b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(31935960a^6b^7+77558760a^7b^7+77189562432b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(49573632864ab^8+80994977424a^2b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(5675445000a^3b^8+5939968500a^4b^8+56728350a^5b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(59879925a^6b^8+2483214240b^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(6999713760ab^9+1263236520a^2b^9+1386606000a^3b^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(32516250a^4b^9+27313650a^5b^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(193882832b^{10}+116414584ab^{10}+163524504a^2b^{10})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(8708700a^3b^{10}+7153575a^4b^{10})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(3327896b^{11}+8390512ab^{11}+1099332a^2b^{11})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(1017900a^3b^{11}+131404b^{12}+58058ab^{12})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536a(71253a^2b^{12}+910b^{13}+2030ab^{13}+15b^{14})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{32768(1428329123020800a+2322150583173120a^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(1606274243887104a^3+636906005299200a^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(163554924216320a^5+29011643781120a^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(3688786669568a^7+343226083200a^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(23578343360a^9+1193992800a^{10}+43995952a^{11})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(1146600a^{12}+20020a^{13}+210a^{14}+a^{15})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(14283291230208000b+8525223163330560a^2b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(2182234506854400a^3b+1969113895649280a^4b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(264966591528960a^5b+82650036080640a^6b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(6233334240000a^7b+920583637440a^8b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(39161191680a^9b+3018684240a^{10}b+67407600a^{11}b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(2691780a^{12}b+24360a^{13}b+435a^{14}b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-2322150583173120b^2+8525223163330560ab^2+5852172660080640a^3b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(528194276167680a^4b^2+497028396625920a^5b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(30401221536000a^6b^2+9824497355520a^7b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(363512823840a^8b^2+54807626640a^9b^2+1104760800a^{10}b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(85096440a^{11}b^2+712530a^{12}b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(27405a^{1}3b^{2}+1606274243887104b^{3})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-2182234506854400ab^{3}+5852172660080640a^{2}b^{3})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(1153266031104000a^{4}b^{3}+46525375584000a^{5}b^{3})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(42437501433600a^{6}b^{3}+1285763673600a^{7}b^{3})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(404597084400a^{8}b^{3}+7227090000a^{9}b^{3}+1052508600a^{10}b^{3})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(8143200a^{11}b^{3}+593775a^{12}b^{3})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-636906005299200b^{4}+1969113895649280ab^{4}-528194276167680a^{2}b^{4})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(1153266031104000a^{3}b^{4}+85800120220800a^{5}b^{4})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(1679965963200a^{6}b^{4}+1438362021600a^{7}b^{4})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(20980485000a^{8}b^{4}+6230113500a^{9}b^{4}+42921450a^{10}b^{4})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(5852925a^{11}b^{4}+163554924216320b^{5})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-264966591528960ab^{5}+497028396625920a^{2}b^{5})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-46525375584000a^{3}b^{5}+85800120220800a^{4}b^{5})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(2667136741920a^6b^5+24708348000a^7b^5+19527158700a^8b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(109254600a^9b^5+30045015a^{10}b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-290116437811120b^6+82650036080640ab^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-30401221536000a^2b^6+42437501433600a^3b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-1679965963200a^4b^6+2667136741920a^5b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(34171477200a^7b^6+119759850a^8b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(86493225a^9b^6+36887866695680b^7-6233334240000ab^7+9824497355520a^2b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-1285763673600a^3b^7+1438362021600a^4b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-24708348000a^5b^7+34171477200a^6b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(145422675a^8b^7-343226083200b^8+920583637440ab^8-363512823840a^2b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(404597084400a^3b^8-20980485000a^4b^8+19527158700a^5b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-119759850a^6b^8+145422675a^7b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(235783433360b^9-39161191680ab^9+54807626640a^2b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-7227090000a^3b^9+6230113500a^4b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{32768(-109254600a^5b^9+86493225a^6b^9-1193992800b^{10})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{32768(3018684240ab^{10}-1104760800a^2b^{10})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{32768(1052508600a^3b^{10}-42921450a^4b^{10}+30045015a^5b^{10})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{32768(43995952b^{11}-67407600ab^{11})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{32768(85096440a^2b^{11}-8143200a^3b^{11}+5852925a^4b^{11})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{32768(-1146600b^{12}+2691780ab^{12}-712530a^2b^{12})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{32768(593775a^3b^{12}+20020b^{13}-24360ab^{13})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{32768(27405a^2b^{13}-210b^{14}+435ab^{14}+b^{15})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}\Bigg\}. \tag{7}$$

§3. Derivation of main summation formula (7)

Substituting $c=\frac{a+b+32}{2}$, and $z=\frac{1}{2}$, in equation (2), we get

$$(a-b)\ {}_2F_1\left[\begin{matrix}a,\ b\\ \frac{a+b+32}{2}\end{matrix};\ \frac{1}{2}\right]=a\ {}_2F_1\left[\begin{matrix}a+1,\ b\\ \frac{a+b+32}{2}\end{matrix};\ \frac{1}{2}\right]-b\ {}_2F_1\left[\begin{matrix}a,\ b+1\\ \frac{a+b+32}{2}\end{matrix};\ \frac{1}{2}\right].$$

Now using Gauss second summation theorem, we get

$$\begin{aligned}&L.\ H.\ S\\ =\ & a\ \frac{2^b\ \Gamma(\frac{a+b+32}{2})}{\Gamma(b)}\Bigg[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+2}{2})}\Bigg\{\frac{16384(1428329123020800a-2322150583173120a^2)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}\\ &+\frac{16384(1606274243887104a^3-636906005299200a^4)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}\end{aligned}$$

$$+\frac{16384(163554924216320a^5-29011643781120a^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(3688786669568a^7-343226083200a^8+23578343360a^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(-1193992800a^{10}+43995952a^{11})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(-1146600a^{12}+20020a^{13}-210a^{14}+a^{15}+14283291230208 00b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(8525223163330560a^2b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(-2182234506854400a^3b+1969113895649280a^4b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(-264966591528960a^5b+82650036080640a^6b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(-6233334240000a^7b+920583637440a^8b-39161191680a^9b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(3018684240a^{10}b-67407600a^{11}b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(2691780a^{12}b-24360a^{13}b+435a^{14}b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(2322150583173120b^2+8525223163330560ab^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(5852172660080640a^3b^2-528194276167680a^4b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(497028396625920a^5b^2-30401221536000a^6b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(9824497355520a^7b^2-363512823840a^8b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(54807626640a^9b^2-1104760800a^{10}b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(85096440a^{11}b^2-712530a^{12}b^2+27405a^{13}b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(1606274243887104b^3+2182234506854400ab^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(5852172660080640a^2b^3+1153266031104000a^4b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(-46525375584000a^5b^3+42437501433600a^6b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(-1285763673600a^7b^3+404597084400a^8b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(-7227090000a^9b^3+1052508600a^{10}b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(-8143200a^{11}b^3+593775a^{12}b^3+636906005299200b^4+1969113895649280ab^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(528194276167680a^2b^4+1153266031104000a^3b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(85800120220800a^5b^4-1679965963200a^6b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(1438362021600a^7b^4-20980485000a^8b^4+6230113500a^9b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{16384(-42921450a^{10}b^4+5852925a^{11}b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(1635549242163 20b^5+264966591528960ab^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(497028396625920a^2b^5+46525375584000a^3b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(85800120220800a^4b^5+2667136741920a^6b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(-24708348000a^7b^5+19527158700a^8b^5-109254600a^9b^5+30045015a^{10}b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(290116437811 20b^6+82650036080640ab^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(30401221536000a^2b^6+42437501433600a^3b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(1679965963200a^4b^6+2667136741920a^5b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(34171477200a^7b^6-119759850a^8b^6+86493225a^9b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(36887866695 68b^7+6233334240000ab^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(9824497355520a^2b^7+1285763673600a^3b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(1438362021600a^4b^7+24708348000a^5b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(34171477200a^6b^7+145422675a^8b^7+343226083200b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(920583637440ab^8+363512823840a^2b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(404597084400a^3b^8+20980485000a^4b^8+19527158700a^5b^8)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(119759850a^6b^8+145422675a^7b^8)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(23578343360b^9+39161191680ab^9+54807626640a^2b^9)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(7227090000a^3b^9+6230113500a^4b^9)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(109254600a^5b^9+86493225a^6b^9+1193992800b^{10})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(3018684240ab^{10}+1104760800a^2b^{10})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(1052508600a^3b^{10}+42921450a^4b^{10}+30045015a^5b^{10})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(43995952b^{11}+67407600ab^{11})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(85096440a^2b^{11}+8143200a^3b^{11}+5852925a^4b^{11})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(1146600b^{12}+2691780ab^{12}+712530a^2b^{12})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{16384(593775a^3b^{12}+20020b^{13}+24360ab^{13})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}\Bigg\}$$

$$+\frac{16384(27405a^2b^{13}+210b^{14}+435ab^{14}+b^{15})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}\Bigg\}$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Bigg\{\frac{65536(1428329123020800-470423898685440a+2328961922629632a^2)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-3811176971304 96a^3+316734590500864a^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-29456251432960a^5+9279610167296a^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-511595950208a^7+77189562432a^8-2483214240a^9)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(193882832a^{10}-3327896a^{11})}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(131404a^{12}-910a^{13}+15a^{14}+4704238986854 40b+5473573561958400ab)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-503214007025664a^2b+1969528317345792a^3b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-150112771248128a^4b+112539365949440a^5b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-5581588819072a^6b+1638717738496a^7b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-49573632864a^8b+6999713760a^9b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-116414584a^{10}b+8390512a^{11}b)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(-58058a^{12}b+2030a^{13}b+2328961922629632b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(503214007025664ab^2+3412315664252928a^2b^2-129681106132992a^3b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(423536993230848a^4b^2-16628085832320a^5b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(11189308499712a^6b^2-300862325376a^7b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(80994977424a^8b^2-1263236520a^9b^2+163524504a^{10}b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-1099332a^{11}b^2+71253a^{12}b^2)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(3811176971304 96b^3+1969528317345792ab^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(1296811061329 92a^2b^3+645921311784960a^3b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-11831795932800a^4b^3+33081548382720a^5b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-686621093760a^6b^3+413821722240a^7b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-5675445000a^8b^3+1386606000a^9b^3-8708700a^{10}b^3)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(1017900a^{11}b^3+316734590500864b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(150112771248128ab^4+423536993230848a^2b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(11831795932800a^3b^4+47032752624000a^4b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-436682232000a^5b^4+1060441956000a^6b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-11019106800a^7b^4+5939968500a^8b^4)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-32516250a^9b^4+7153575a^{10}b^4+294562514329 60b^5+112539365949440ab^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(16628085832320a^2b^5+33081548382720a^3b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(436682232000a^4b^5+1443061650240a^5b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(-6518297520a^6b^5+13870291680a^7b^5-56728350a^8b^5)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(27313650a^9b^5+9279610167296b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(5581588819072ab^6+11189308499712a^2b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(686621093760a^3b^6+1060441956000a^4b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(6518297520a^5b^6+18331241040a^6b^6-31935960a^7b^6)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(59879925a^8b^6+5115959502208b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(1638717738496ab^7+300862325376a^2b^7+413821722240a^3b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(11019106800a^4b^7)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(13870291680a^5b^7+31935960a^6b^7+77558760a^7b^7+77189562432b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(49573632864ab^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$
$$+\frac{65536(80994977424a^2b^8+5675445000a^3b^8+5939968500a^4b^8)}{\Big(\prod\limits_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod\limits_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(56728350a^5b^8+59879925a^6b^8)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(2483214240b^9+6999713760ab^9+1263236520a^2b^9)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(1386606000a^3b^9+32516250a^4b^9)}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(27313650a^5b^9+193882832b^{10}+116414584ab^{10})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(163524504a^2b^{10}+8708700a^3b^{10})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(7153575a^4b^{10}+3327896b^{11}+8390512ab^{11})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(+1099332a^2b^{11}+1017900a^3b^{11}+131404b^{12})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}$$

$$+\frac{65536(58058ab^{12}+71253a^2b^{12}+910b^{13}+2030ab^{13}+15b^{14})}{\Big(\prod_{\zeta=0}^{14}\{a-b-2\zeta\}\Big)\Big(\prod_{\eta=1}^{15}\{a-b+2\eta\}\Big)}\Bigg\}\Bigg]$$

$$-b\,\frac{2^{b+1}\,\Gamma(\frac{a+b+32}{2})}{\Gamma(b+1)}\Bigg[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\,\Bigg\{\frac{16384(1428329123020800a+2322150583173120a^2)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(1606274243887104a^3)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(636906005299200a^4+163554924216320a^5)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(29011643781120a^6+3688786669568a^7)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(343226083200a^8+23578343360a^9+1193992800a^{10})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(43995952a^{11}+1146600a^{12}+20020a^{13})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(210a^{14}+a^{15}+1428329123020800b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(8525223163330560a^2b+2182234506854400a^3b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(1969113895649280a^4b+264966591528960a^5b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(82650036080640a^6b+6233334240000a^7b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(920583637440a^8b+39161191680a^9b+3018684240a^{10}b+67407600a^{11}b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(2691780a^{12}b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(24360a^{13}b+435a^{14}b-23221505831731 20b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(8525223163330560ab^2+5852172660080640a^3b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(528194276167680a^4b^2+497028396625920a^5b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(30401221536000a^6b^2+9824497355520a^7b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(363512823840a^8b^2+54807626640a^9b^2+1104760800a^{10}b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(+85096440a^{11}b^2+712530a^{12}b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(27405a^{13}b^2+1606274243887104b^3)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-2182234506854400ab^3+5852172660080640a^2b^3)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(1153266031104000a^4b^3+46525375584000a^5b^3)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(42437501433600a^6b^3+1285763673600a^7b^3)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(404597084400a^8b^3+7227090000a^9b^3+1052508600a^{10}b^3)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(8143200a^{11}b^3+593775a^{12}b^3)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-636906005299200b^4+1969113895649280ab^4-528194276167680a^2b^4)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(1153266031104000a^3b^4+85800120220800a^5b^4)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(1679965963200a^6b^4+1438362021600a^7b^4)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(20980485000a^8b^4+6230113500a^9b^4+42921450a^{10}b^4)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(5852925a^{11}b^4+163554924216320b^5)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(2667136741920a^6b^5+24708348000a^7b^5+19527158700a^8b^5)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(109254600a^9b^5+30045015a^{10}b^5)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-290116437811120b^6+82650036080640ab^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-30401221536000a^2b^6+42437501433600a^3b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-1679965963200a^4b^6+2667136741920a^5b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(34171477200a^7b^6+119759850a^8b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(86493225a^9b^6+36887866695680b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-6233334240000ab^7+9824497355520a^2b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-1285763673600a^3b^7+1438362021600a^4b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-24708348000a^5b^7+34171477200a^6b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(145422675a^8b^7-343226083200b^8+920583637440ab^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-363512823840a^2b^8+404597084400a^3b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-20980485000a^4b^8+19527158700a^5b^8-119759850a^6b^8+145422675a^7b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(235783433360b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{16384(-39161191680ab^9+54807626640a^2b^9-7227090000a^3b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(6230113500a^4b^9-109254600a^5b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(86493225a^6b^9-1193992800b^{10}+3018684240ab^{10})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-1104760800a^2b^{10}+1052508600a^3b^{10})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-42921450a^4b^{10}+30045015a^5b^{10}+43995952b^11)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-67407600ab^{11}+85096440a^2b^{11})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-8143200a^3b^{11}+5852925a^4b^{11}-1146600b^{12}+2691780ab^{12})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(-712530a^2b^{12}+593775a^3b^{12})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{16384(20020b^{13}-24360ab^{13}+27405a^2b^{13}-210b^14+435ab^{14}+b^{15})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}\Bigg\}-\frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}$$

$$\times\Bigg\{\frac{65536(1428329123020800+470423898685440a+2328961922629632a^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(381117697130496a^3+316734590500864a^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(29456251432960a^5+9279610167296a^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(511595950208a^7+77189562432a^8+2483214240a^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(193882832a^{10}+3327896a^{11})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(131404a^{12}+910a^{13}+15a^{14}-470423898685440b+5473573561958400ab)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(503214007025664a^2b+1969528317345792a^3b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(150112771248128a^4b+112539365949440a^5b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(5581588819072a^6b+1638717738496a^7b+49573632864a^8b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(6999713760a^9b+116414584a^{10}b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(8390512a^{11}b+58058a^{12}b+2030a^{13}b)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(2328961922629632b^2-503214007025664ab^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(3412315664252928a^2b^2+129681106132992a^3b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(423536993230848a^4b^2+16628085832320a^5b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(111893084997 12a^6b^2+300862325376a^7b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(80994977424a^8b^2+1263236520a^9b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(163524504a^{10}b^2+1099332a^{11}b^2+71253a^{12}b^2)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-3811176971304 96b^3+1969528317345792ab^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-129681106132992a^2b^3+645921311784960a^3b^3+11831795932800a^4b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(33081548382720a^5b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(686621093760a^6b^3+413821722240a^7b^3+5675445000a^8b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(1386606000a^9b^3+8708700a^{10}b^3)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(1017900a^{11}b^3+3167345905008864b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-150112771248128ab^4+423536993230848a^2b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-11831795932800a^3b^4+47032752624000a^4b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(436682232000a^5b^4+1060441956000a^6b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(11019106800a^7b^4+5939968500a^8b^4+32516250a^9b^4)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(7153575a^{10}b^4-294562514329 60b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(112539365949440ab^5-16628085832320a^2b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(33081548382720a^3b^5-436682232000a^4b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(1443061650240a^5b^5+6518297520a^6b^5+13870291680a^7b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(56728350a^8b^5+27313650a^9b^5)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(9279610167296b^6-5581588819072ab^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(11189308499712a^2b^6-686621093760a^3b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(1060441956000a^4b^6-6518297520a^5b^6+18331241040a^6b^6+31935960a^7b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(59879925a^8b^6)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-511595950208b^7+1638717738496ab^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-300862325376a^2b^7+413821722240a^3b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-11019106800a^4b^7+13870291680a^5b^7-31935960a^6b^7)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(77558760a^7b^7+77189562432b^8+5939968500a^4b^8-56728350a^5b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-49573632864ab^8+80994977424a^2b^8-5675445000a^3b^8)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(59879925a^6b^8-2483214240b^9+6999713760ab^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-1263236520a^2b^9+1386606000a^3b^9)}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-32516250a^4b^9+27313650a^5b^9+193882832b^{10})}{\Big(\prod\limits_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod\limits_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$

$$+\frac{65536(-116414584ab^{10}+163524504a^2b^{10})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{65536(-8708700a^3b^{10}+7153575a^4b^{10}-3327896b^11)}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{65536(8390512ab^{11}-1099332a^2b^{11}+1017900a^3b^{11})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}$$
$$+\frac{65536(131404b^{12}-58058ab^{12}+71253a^2b^{12}-910b^13+2030ab^{13}+15b^{14})}{\Big(\prod_{\alpha=0}^{15}\{a-b-2\alpha\}\Big)\Big(\prod_{\beta=1}^{14}\{a-b+2\beta\}\Big)}\Bigg\}\Bigg].$$

On simplification the result (8) is proved.

References

[1] Arora, Asish, Singh, Rahul and Salahuddin, Development of a family of summation formulae of half argument using Gauss and Bailey theorems, Journal of Rajasthan Academy of Physical Sciences, **7**(2008), 335-342.

[2] J. L. Lavoie, Some summation formulae for the series ${}_3F_2$, Math. Comput., **49**(1987), 269-274.

[3] J. L. Lavoie, F. Grondin and A. K. Rathie, Generalizations of Watson's theorem on the sum of a ${}_3F_2$, Indian J. Math., **34**(1992), 23-32.

[4] J. L. Lavoie, F. Grondin and A. K. Rathie, Generalizations of Whipple's theorem on the sum of a ${}_3F_2$, J. Comput. Appl. Math., **72**(1996), 293-300.

[5] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, More Special Functions, Integrals and Series, **3**(1986).

[6] E. D. Rainville, The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n\ (\zeta, p, \nu)$, Bull. Amer. Math. Soc., **51**(1945), 714-723.

[7] Salahuddin and M. P Chaudhary, Development of some summation formulae using Hypergeometric function, Global journal of Science Frontier Research, **10**(2010), 36-48.

[8] Salahuddin and M. P Chaudhary, Certain summation formulae associated to Gauss second summation theorem, Global journal of Science Frontier Research, **10**(2010), 30-35.

Scientia Magna
Vol. 7 (2011), No. 4, 82-91

Some classes of difference fuzzy numbers defined by an orlicz function

Ayhan Esi[†] and Eda Eren[‡]

Department of Mathematics, Adiyaman University,
02040, Adiyaman, Turkey
E-mail: aesi23@hotmail.com eeren@adiyaman.edu.tr

Abstract In this paper we introduce some new classes of sequences of fuzzy numbers using by Orlicz function and examine some properties of resulting sequence classes of fuzzy numbers.

Keywords Fuzzy numbers, fuzzy sets, difference sequence, Orlicz function.
2000 Mathematics Subject Classification: Primary 05C38, 15A15, Secondary 05A15, 1 5A18.

§1. Introduction and preliminaries

The notion of fuzzy sets was introduced by Zadeh [7] in 1965. Afterwards many authors have studied and generalized this notion in many ways, due to potential of the introduced notion. Also it has wide range of applications in almost all the branches of studied in particular science, where mathematics in used. It attracted many workers to introduce different types of classes of fuzzy numbers. Matloka [5] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on, sequences of fuzzy numbers have been discussed by Nanda [6], Diamond and Kloeden [2], Esi [3] and many others.

Let D denote the set of all closed and bounded intervals on $\mathbb{R}$, the real line. For $X,\ Y \in D$, we define

$$\delta\left(X,\ Y\right) = \max\left(\left|a_1 - b_1\right|,\ \left|a_2 - b_2\right|\right),$$

where $X = [a_1,\ a_2]$ and $Y = [b_1,\ b_2]$. It is known that $(D,\ \delta)$ is a complete metric space. A fuzzy real number X is a fuzzy set on $\mathbb{R}$, i.e., a mapping $X : \mathbb{R} \to I\,(= [0,\ 1])$ associating each real number t with its grade of membership $X\,(t)$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $L(\mathbb{R})$. Throughout the paper, by a fuzzy real number X, we mean that $X \in L(\mathbb{R})$.

The $\alpha-$cut or $\alpha-$level set $[X]^{\alpha}$ of the fuzzy real number X, for $0 < \alpha \leq 1$, defined by $[X]^{\alpha} = \{t \in \mathbb{R} :\ X\,(t) \geq \alpha\}$; for $\alpha = 0$, it is the closure of the strong $0-$cut, i.e., closure of the set $\{t \in R :\ X\,(t) > 0\}$. The linear structure of $L(\mathbb{R})$ induces the addition $X + Y$ and the scalar multiplication μX, $\mu \in \mathbb{R}$, in terms of $\alpha-$level sets, by

$$[X + Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha},\ [\mu X]^{\alpha} = \mu\,[X]^{\alpha}$$

for each $\alpha \in (0,\ 1]$.

Let $d : L(\mathbb{R}) \times L(\mathbb{R}) \to R$ be defined by

$$d\,(X,\ Y) = \sup_{0\le\alpha\le 1} \delta\left([X]^{\alpha},\ [Y]^{\alpha}\right).$$

Then d defines a metric on $L(\mathbb{R})$. It is well known that $L(\mathbb{R})$ is complete with respect to d.

The set $\mathbb{R}$ of real numbers can be embedded in $L(\mathbb{R})$ if we define $\overline{r} \in L(\mathbb{R})$ by

$$\overline{r}\,(t) = \begin{cases} 1, & \text{if } t = r\ ; \\ 0, & \text{if } t \neq r\ . \end{cases}$$

The additive identity and multiplicative identity of $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively. For r in $\mathbb{R}$ and X in $L(\mathbb{R})$, the product rX is defined as follows

$$rX\,(t) = \begin{cases} X\left(r^{-1}t\right), & \text{if } r \neq 0\ ; \\ 0, & \text{if } r = 0\ . \end{cases}$$

A metric on $L\,(\mathbb{R})$ is said to be translation invariant if $d\,(X+Z,\ Y+Z) = d\,(X,\ Y)$ for all $X,\ Y,\ Z \in L\,(\mathbb{R})$. The metric d has the following properties:

$$d\,(cX,\ cY) = |c|\,d\,(X,\ Y)$$

for $c \in \mathbb{R}$ and

$$d\,(X+Y,\ Z+W) \le d\,(X,\ Z) + d\,(Y,\ W)\,.$$

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers in to $L(\mathbb{R})$. The fuzzy number X_k denotes the value of the function at $k \in \mathbb{N}$ and is called the $k-$th term of the sequence. We denote by w^F the set of all sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded [5]. We denote by ℓ_∞^F the set of all bounded sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for every $\varepsilon > 0$ there is a positive integer k_0 such that $\overline{d}\,(X_k,\ X_0) < \varepsilon$ for all $k > k_0$ [5]. We denote by c^F the set of all convergent sequences $X = (X_k)$ of fuzzy numbers.

The following inequality will be used throughout the paper: Let $p = (p_k)$ be a positive sequence of real numbers with $0 < \inf p_k = h \le p_k \le \sup_k p_k = H < \infty$ and $K = \max(1,\ 2^{H-1})$. Then for $a_k,\ b_k \in \mathbb{C}$, we have

$$|a_k + b_k|^{p_k} \le K(|a_k|^{p_k} + |b_k|^{p_k}),\ \text{ for all } k \in \mathbb{N}. \tag{1}$$

An Orlicz function is a function $M : [0,\ \infty) \to [0,\ \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function is a function M is said to satisfy the Δ_2-condition for all values of x, if there exist a constant $T > 0$ such that $M(2x) \le TM(x)$, $(x \ge 0)$.

Lindenstrauss and Tzafriri [4] used the Orlicz function and introduced the sequence ℓ_M as follows:

$$\ell_M = \left\{ x = (x_k) : \sum_k M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|x\| = \inf\left\{ r > 0 : \sum_k M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

Remark 1.1. An Orlicz function M satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

In this paper we define the following difference classes of fuzzy numbers: Let $X = (X_k)$ be a sequence of fuzzy numbers and M be an Orlicz function. In this paper and define the following difference classes of fuzzy numbers:

$$c_0^F[M,\ \Delta,\ p,\ s] = \left\{ X = (X_k) \in w^F : \lim_k k^{-s} \left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho}\right)\right]^{p_k} = 0 \right.$$
$$\left. \text{for some } \rho > 0,\ s \geqslant 0 \right\},$$

$$c^F[M,\ \Delta,\ p,\ s] = \left\{ X = (X_k) \in w^F \ : \lim_k k^{-s} \left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} = 0 \right.$$
$$\left. \text{for some } \rho > 0,\ s \geqslant 0 \right\}$$

and

$$\ell_\infty^F[M,\ \Delta,\ p,\ s] = \left\{ X = (X_k) \in w^F : \sup_k k^{-s} \left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho}\right)\right]^{p_k} < \infty \right.$$
$$\left. \text{for some } \rho > 0,\ s \geqslant 0 \right\},$$

where $p = (p_k)$ is a sequence of real numbers such that $p_k > 0$ for all $k \in \mathbb{N}$ and $\sup_k p_k = H < \infty$ and $\Delta X_k = X_k - X_{k+1}$, this assumption is made throughout the rest of this paper.

We can some specialize these classes as follows: If $s = 0$, $M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, then

$$c_0^F[M,\ \Delta,\ p,\ s] = c_0^F[\Delta] = \left\{ X = (X_k) \in w^F : \lim_k d(\Delta X_k,\ \overline{0}) = 0 \right\},$$

$$c^F[M,\ \Delta,\ p,\ s] = c^F[\Delta] = \left\{ X = (X_k) \in w^F : \lim_k\ d(\Delta X_k,\ X) = 0 \right\}$$

and

$$\ell_\infty^F[M,\ \Delta,\ p,\ s] = \ell_\infty^F[\Delta] = \left\{ X = (X_k) \in w^F : \sup_k d(\Delta X_k,\ \overline{0}) < \infty \right\},$$

which were defined and studied by Başarır and Mursaleen [1].

If $s = 0$, $M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$ and $\Delta X_k = X_k$ then we obtain the classes c_0^F, c^F and ℓ_∞^F which we defined and studied by Nanda [6].

Example 1.2. Let $s = 0$, $M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence of fuzzy numbers $X = (X_k)$ defined as follows:

$$\text{For } k = i^2,\ \ i \in \mathbb{N},\ \ X_k = \overline{0}.$$

$$\text{Otherwise, } X_k(t) = \begin{cases} \overline{1}, & \text{for } 0 \leq t \leq k^{-1}; \\ \overline{0}, & \text{otherwise}, \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [0,\ 0], & \text{for } k = i^2,\ i \in \mathbb{N}; \\ \left[0,\ k^{-1}\right], & \text{otherwise}, \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[-(k+1)^{-1},\ 0\right], & \text{for } k = i^2,\ i \in \mathbb{N}; \\ \left[0,\ k^{-1}\right], & \text{for } k = i^2 - 1,\ i \in \mathbb{N}, \text{ with } i > 1; \\ \left[-(k+1)^{-1},\ k^{-1}\right], & \text{otherwise}, \end{cases}$$

Hence $X = (X_k) \in c_0^F[M, \Delta,\ p,\ s] \subset c^F[M,\ \Delta,\ p,\ s] \subset \ell_\infty^F[M,\ \Delta,\ p,\ s]$.

§2. Main results

In this section we prove some results involving the classes of sequences of fuzzy numbers $c_0^F[M,\ \Delta,\ p,\ s]$, $c^F[M,\ \Delta,\ p,\ s]$ and $\ell_\infty^F[M,\ \Delta,\ p,\ s]$.

Theorem 2.1. If d is a translation invariant metric, then $c_0^F[M, \Delta,\ p,\ s]$, $c^F[M,\ \Delta,\ p,\ s]$ and $\ell_\infty^F[M,\ \Delta,\ p,\ s]$ are closed under the operations of addition and scalar multiplication.

Proof. Since d is a translation invariant metric implies that

$$d(\Delta X_k + \Delta Y_k,\ \overline{0}) \leqslant d(\Delta X_k,\ \overline{0}) + d(\Delta Y_k,\ \overline{0}) \tag{2}$$

and

$$d(\lambda \Delta X_k,\ \overline{0}) = |\lambda|\, d(\Delta X_k,\ \overline{0}), \tag{3}$$

where λ is a scalar. Let $X = (X_k)$ and $Y = (Y_k) \in \ell_\infty^F[M,\ \Delta,\ p,\ s]$. Then there exists positive numbers ρ_1 and ρ_2 such that

$$\sup_k\ k^{-s}\left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho_1}\right)\right]^{p_k} < \infty$$

and

$$\sup_k k^{-s}\left[M\left(\frac{d(\Delta Y_k,\ \overline{0})}{\rho_2}\right)\right]^{p_k} < \infty.$$

Let $\rho_3 = \max(2\rho_1,\ 2\rho_2)$. By taking into account the properties (1), (2) and since M is non-decreasing and convex function, we have

$$\begin{aligned} k^{-s}\left[M\left(\frac{d(\Delta X_k+\Delta Y_k,\ \overline{0})}{\rho_3}\right)\right]^{p_k} &\leq k^{-s}\left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho_3}\right)+M\left(\frac{d(\Delta Y_k,\ \overline{0})}{\rho_3}\right)\right]^{p_k} \\ &\leq k^{-s}\left[\frac{1}{2}M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho_1}\right)+\frac{1}{2}M\left(\frac{d(\Delta Y_k,\ \overline{0})}{\rho_2}\right)\right]^{p_k} \\ &\leq Kk^{-s}\left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho_1}\right)\right]^{p_k}+Kk^{-s}\left[M\left(\frac{d(\Delta Y_k,\ \overline{0})}{\rho_2}\right)\right]^{p_k} \\ &< \infty. \end{aligned}$$

Therefore $X+Y \in \ell_\infty^F[M,\ \Delta,\ p,\ s]$. Now let $X=(X_k) \in \ell_\infty^F[M,\ \Delta,\ p,\ s]$ and $\lambda \in \mathbb{R}$ with $0<|\lambda|<1$. By taking into account the properties (3) and Remark, we have

$$\begin{aligned} k^{-s}\left[M\left(\frac{d(\lambda\Delta X_k,\ \overline{0})}{\rho}\right)\right]^{p_k} &\leqslant k^{-s}\left[|\lambda|\,M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho}\right)\right]^{p_k} \\ &\leqslant k^{-s}\,|\lambda|^{p_k}\left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho}\right)\right]^{p_k} \\ &\leqslant |\lambda|^{h}\,k^{-s}\left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho}\right)\right]^{p_k} \\ &< \infty. \end{aligned}$$

Therefore $\lambda X \in \ell_\infty^F[M,\ \Delta,\ p,\ s]$. The proof of the other cases are routine work in view of the above proof.

Theorem 2.2. Let M be an Orlicz function, then

$$c_0^F[M,\ \Delta,\ p,\ s] \subset c^F[M,\ \Delta,\ p,\ s] \subset \ell_\infty^F[M,\ \Delta,\ p,\ s].$$

Proof. Clearly $c_0^F[M,\ \Delta,\ p,\ s] \subset c^F[M,\ \Delta,\ p,\ s]$. We shall prove that $c^F[M,\ \Delta,\ p,\ s] \subset \ell_\infty^F[M,\ \Delta,\ p,\ s]$. Let $X=(X_k) \in c^F[M,\ \Delta,\ p,\ s]$. Then there exists some fuzzy number X_0 such that

$$\lim_k k^{-s}\left[M\left(\frac{d(\Delta X_k,\ X_o)}{\rho}\right)\right]^{p_k}=0,\ s\geqslant 0.$$

On taking $\rho_1=2\rho$ and by using (1), we have

$$\begin{aligned} k^{-s}\left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho_1}\right)\right]^{p_k} \leqslant\ & \left(\frac{1}{2}\right)^{p_k} Kk^{-s}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} \\ &+\left(\frac{1}{2}\right)^{p_k} Kk^{-s}\left[M\left(\frac{d(\Delta X_k,\ \overline{0})}{\rho}\right)\right]^{p_k}. \end{aligned}$$

Since $X=(X_k) \in c^F[M,\ \Delta,\ p,\ s]$, we get $X=(X_k) \in \ell_\infty^F[M,\ \Delta,\ p,\ s]$.

Theorem 2.3. $c_0^F[M,\ \Delta,\ p,\ s]$, $c^F[M,\ \Delta,\ p,\ s]$ and $\ell_\infty^F[M,\ \Delta,\ p,\ s]$ are complete metric spaces with respect to the metric given by

$$g(X,\ Y)=\inf\left\{\rho^{\frac{pn}{H}}>0:\ \left\{\sup_k\left[k^{-s}\left(M\left(\frac{(d(\Delta X_k,\ \Delta Y_k)}{\rho}\right)\right)\right]^{p_k}\right\}^{\frac{1}{H}}\leq 1\right\},$$

where $H = \max(1,\ \sup_k p_k)$.

Proof. Consider the class $c_0^F[M,\ \Delta,\ p,\ s]$. Let $\left(X^i\right)$ be a Cauchy sequence in $c_0^F[M,\ \Delta,\ p,\ s]$. Let $\varepsilon > 0$ be given and $r,\ x_o > 0$ be fixed. Then for each $\frac{\varepsilon}{rx_o} > 0$, there exists a positive integer n_o such that

$$g\left(X^i,\ X^j\right) < \frac{\varepsilon}{rx_o}, \text{ for all } i,\ j \geq n_o.$$

By definition of g, we have

$$\left\{\sup_k \left[k^{-s}\left(M\left(\frac{d(\Delta X_k^i,\ \Delta X_k^j)}{g\left(X^i,\ X^j\right)}\right)\right)\right]^{p_k}\right\}^{\frac{1}{H}} \leq 1,$$

for all $i,\ j \geq n_o$. Thus

$$\sup_k \left[k^{-s}\left(M\left(\frac{d(\Delta X_k^i,\ \Delta X_k^j)}{g\left(X^i,\ X^j\right)}\right)\right)\right]^{p_k} \leq 1,$$

for all $i,\ j \geq n_o$. It follows that

$$k^{-s}\left(M\left(\frac{d(\Delta X_k^i,\ \Delta X_k^j)}{g\left(X^i,\ X^j\right)}\right)\right) \leq 1,$$

for each $k \geq 0$ and for all $i,\ j \geq n_o$. For $r > 0$ with $k^{-s}M\left(\frac{rx_o}{2}\right) \geq 1$, we have

$$k^{-s}\left(M\left(\frac{d(\Delta X_k^i,\ \Delta Y_k^j)}{g\left(X^i,\ Y^j\right)}\right)\right) \leq k^{-s}M\left(\frac{rx_o}{2}\right).$$

Since $k^{-s} \neq 0$ for all k, we have

$$M\left(\frac{d(\Delta X_k^i,\ \Delta X_k^j)}{g\left(X^i,\ X^j\right)}\right) \leq M\left(\frac{rx_o}{2}\right).$$

This implies that

$$d(\Delta X_k^i,\ \Delta X_k^j) \leq \frac{rx_o}{2}.\frac{\varepsilon}{rx_o} = \frac{\varepsilon}{2}\ .$$

Hence $\left(\Delta X_k^i\right)_i = \left(\Delta X_k^1,\ \Delta X_k^2,\ \Delta X_k^3,\ \cdots\right)$ is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete, it is convergent. Therefore for each $\varepsilon\ (0 < \varepsilon < 1)$, there exists a positive integer n_o such that $d(\Delta X_k^i,\ \Delta X_k^j) < \varepsilon$ for all $i \geq n_o$. Using the continuity of the Orlicz function M, we can find

$$\left\{\sup_{k \geq n_o} \left[k^{-s}\left(M\left(\frac{\lim_{j\to\infty} d(\Delta X_k^i,\ \Delta X_k^j)}{\rho}\right)\right)\right]^{p_k}\right\}^{\frac{1}{H}} \leq 1.$$

Thus

$$\left\{\sup_{k \geq n_o} \left[k^{-s}\left(M\left(\frac{\lim_{j\to\infty} d(\Delta X_k^i,\ \Delta X_k)}{\rho}\right)\right)\right]^{p_k}\right\}^{\frac{1}{H}} \leq 1.$$

Taking infimum of such $\rho's$ we get

$$\inf\left\{\rho^{\frac{pn}{H}} > 0 : \left\{\sup_k \left[k^{-s}\left(M\left(\frac{\left(d(\Delta X_k^i,\ \Delta X_k)\right)}{\rho}\right)\right)\right]^{p_k}\right\}^{\frac{1}{H}} \leq 1\right\} < \varepsilon,$$

for all $i \geq n_o$. That is $X^i \to X$ in $c_0^F[M,\ \Delta,\ p,\ s]$, where $X = (X_k)$. Now let $i \geq n_o$, then using triangular inequality, we get

$$g\left(X,\ \overline{0}\right) \leq g\left(X,\ X^i\right) + g\left(X^i,\ \overline{0}\right).$$

So we have $X \in c_0^F[M,\ \Delta,\ p,\ s]$. This completes the proof of the theorem.

Theorem 2.4. Let $\inf p_k = h > 0$ and $\sup_k p_k = H < \infty$. Then

(a) $c^F[M,\ \Delta] \subset c^F[M,\ \Delta,\ p,\ s]$.

(b) Let M be an Orlicz function which satisfies Δ_2-condition. Then $c^F[\Delta,\ p,\ s] \subset c^F[M,\ \Delta,\ p,\ s]$.

Proof. (a) Suppose that $X = (X_k) \in c^F[M,\ \Delta]$. Since M is an Orlicz function, then

$$\lim_k M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right) = M\left[\lim_k \frac{d(\Delta X_k,\ X_0)}{\rho}\right] = 0,\ \text{ for some } \rho > 0.$$

Since $\inf_k p_k = h > 0$, then $\lim_k \left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^h = 0$, so for $0 < \varepsilon < 1$, $\exists\ k_0 \ni$ for all $k > k_0$,

$$\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^h < \varepsilon < 1$$

and since $p_k \geqslant h$ for all $k \in \mathbb{N}$

$$\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} \leqslant \left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^h < \varepsilon < 1$$

and then we get

$$\lim_k \left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} = 0.$$

Since (k^{-s}) is bounded, we write

$$\lim_k k^{-s}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} = 0.$$

Therefore $X = (X_k) \in c^F[M,\ \Delta,\ p,\ s]$.

(b) Let $X = (X_k) \in c^F[\Delta,\ p,\ s]$ so that $S_k = k^{-s}\left[d(\Delta X_k,\ X_0)\right]^{p_k} \longrightarrow 0$ as $k \longrightarrow \infty$. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Now we write $y_k = \frac{d(\Delta X_k,\ X_0)}{\rho}$ for all $k \in \mathbb{N}$ and consider

$$k^{-s}\left[M(y_k)\right]^{p_k} = k^{-s}\left[M(y_k)\right]^{p_k}\ |_{y_k \leq \delta} + k^{-s}\left[M(y_k)\right]^{p_k}\ |_{y_k > \delta}\ .$$

Using by the remark we have $k^{-s}\left[M(y_k)\right]^{p_k} \leq k^{-s}\max(\varepsilon,\ \varepsilon^h)$ for $y_k \leq \delta$. For $y_k > \delta$, we will make the following procedure. We have

$$y_k \leq \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.$$

Since M is non-decreasing and convex, it follows that

$$M(y_k) < M\left(1 + \frac{y_k}{\delta}\right) \leq \frac{1}{2}M(2) + \frac{1}{2}(2y_k\delta^{-1})\ .$$

Since M satisfies Δ_2-condition, we can write

$$M(y_k) = \frac{T}{2}\frac{y_k}{\delta}M(2) + \frac{T}{2}\frac{y_k}{\delta}M(2) = T\frac{y_k}{\delta}M(2).$$

We get the following

$$k^{-s}\left[M(y_k)\right]^{p_k}_{y_k>\delta} \leq k^{-s}\max\left(1,\ \left[TM(2)\delta^{-1}\right]^H [y_k]^{p_k}\right)$$

and

$$k^{-s}\left[M(y_k)\right]^{p_k} \leq k^{-s}\max(\varepsilon,\ \varepsilon^h) + \max\left(1,\ \left[T\delta^{-1}M(2)\right]^H\right) S_k.$$

Taking the limits $\varepsilon \to 0$ and $k \to \infty$, it follows that $X = (X_k) \in c^F\left[M,\ \triangle,\ p,\ s\right]$.

Theorem 2.5. Let M, M_1 and M_2 be Orlicz functions and $s_1,\ s_2 \geq 0$. Then

(a) $c^F\left[M_1,\ \triangle,\ p,\ s\right]\bigcap c^F\left[M_2,\ \triangle,\ p,\ s\right] \subset c^F\left[M_1+M_2,\ \triangle,\ p,\ f,\ s\right]$.

(b) $s_1 \leq s_2$ implies $c^F\left[M,\ \triangle,\ p,\ s_1\right] \subset c^F\left[M,\ \triangle,\ p,\ s_2\right]$.

Proof. (a) Let $X = (X_k) \in c^F\left[M_1,\ \triangle,\ p,\ s\right]\bigcap c^F\left[M_2,\ \triangle,\ p,\ s\right]$. Then there exist some $\rho_1,\ \rho_2 > 0$ such that

$$\lim_k k^{-s}\left[M_1\left(\frac{d(\Delta X_k,\ X_0)}{\rho_1}\right)\right]^{p_k} = 0$$

and

$$\lim_k k^{-s}\left[M_2\left(\frac{d(\Delta X_k,\ X_0)}{\rho_2}\right)\right]^{p_k} = 0.$$

Let $\rho = \rho_1 + \rho_2$. Then using the inequality (1), we have

$$\begin{aligned}
\left[(M_1+M_2)\left[\frac{d(\Delta X_k,\ X_0)}{\rho}\right]\right]^{p_k} &= \left[M_1\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right) + M_2\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} \\
&\leq \left[\frac{\rho_1}{\rho_1+\rho_2}M_1\left(\frac{d(\Delta X_k,\ X_0)}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2}M_2\left(\frac{d(\Delta X_k,\ X_0)}{2\rho_2}\right)\right]^{p_k} \\
&\leq \left[M_1\left(\frac{d(\Delta X_k,\ X_0)}{\rho_1}\right) + M_2\left(\frac{d(\Delta X_k,\ X_0)}{\rho_2}\right)\right]^{p_k} \\
&\leq K\left[M_1\left(\frac{d(\Delta X_k,\ X_0)}{\rho_1}\right)\right]^{p_k} + K\left[M_2\left(\frac{d(\Delta X_k,\ X_0)}{\rho_2}\right)\right]^{p_k}.
\end{aligned}$$

Since (k^{-s}) is bounded, we write

$$\begin{aligned}
k^{-s}\left[(M_1+M_2)\left[\frac{d(\Delta X_k,\ X_0)}{\rho}\right]\right]^{p_k} \leq\ & K\ k^{-s}\left[M_1\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} \\
& + K\ k^{-s}\left[M_2\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k}.
\end{aligned}$$

Therefore we get $X = (X_k) \in c^F\left[M_1+M_2,\ \triangle,\ p,\ f,\ s\right]$.

(b) Let $s_1 \leq s_2$. Then $k^{-s_2} \leq k^{-s_1}$ for all $k \in \mathbb{N}$. Let $X = (X_k) \in c^F\left[M,\ \Delta,\ p,\ s_1\right]$. Since

$$k^{-s_2}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} \leq\ k^{-s_1}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k},$$

we get $X = (X_k) \in c^F\left[M,\ \triangle,\ p,\ s_2\right]$.

Theorem 2.6. Let M be an Orlicz function. Then

(a) Let $0 < \inf_k p_k \leq p_k \leq 1$, then $c^F\,[M,\ \Delta,\ p,\ s] \subset c^F\,[M,\ \Delta,\ s]$.

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$, then $c^F\,[M,\ \Delta,\ s] \subset c^F\,[M,\ \Delta,\ p,\ s]$.

(c) Let $0 < p_k \leq q_k$ and $\left(\frac{q_k}{p_k}\right)$ be bounded, then $c^F\,[M,\ \Delta,\ q,\ s] \subset c^F\,[M,\ \Delta,\ p,\ s]$.

Proof. (a) $X = (X_k) \in c^F\,[M,\ \triangle,\ p,\ s]$. Since $0 < \inf_k p_k \leq p_k \leq 1$, for each k, we get

$$k^{-s}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right] \leq k^{-s}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k}$$

for all $k \in \mathbb{N}$, and for some $\rho > 0$. So we get $X = (X_k) \in c^F\,[M,\ \Delta,\ s]$.

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$, for each k and $X = (X_k) \in c^F\,[M,\ \Delta,\ s]$. Then for each $0 < \varepsilon < 1$ there exists a positive integer k_0 such that

$$k^{-s}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right] \leq \varepsilon < 1$$

for each $k \geq k_0$. This implies that

$$k^{-s}\left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{p_k} \leq k^{-s}M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)$$

for some $\rho > 0$. Hence we get $X = (X_k) \in c^F\,[M,\ \Delta,\ p,\ s]$.

(c) Let $X = (X_k) \in c^F\,[M,\ \Delta,\ q,\ s]$. Write

$$w_k = \left[M\left(\frac{d(\Delta X_k,\ X_0)}{\rho}\right)\right]^{q_k}$$

and $T_k = \frac{p_k}{q_k}$, so that $0 < T < T_k \leq 1$ for each k.

We define the sequences (u_k) and (v_k) as follows:

Let $u_k = w_k$ and $v_k = 0$ if $w_k \geq 1$, and let $u_k = 0$ and $w_k = v_k$ if $w_k < 1$.

Then it is clear that for each $k \in \mathbb{N}$, we have

$$w_k = u_k + v_k\ ,\ w_k^{T_k} = u_k^{T_k} + v_k^{T_k}.$$

Now it follows that

$$u_k^{T_k} \leq u_k \leq w_k \text{ and } v_k^{T_k} \leq v_k^T.$$

Therefore

$$k^{-s}w_k^{T_k} = k^{-s}\left[u_k^{T_k} + v_k^{T_k}\right] \leq k^{-s}w_k + k^{-s}v_k^T.$$

Since $T < 1$ so that $T^{-1} > 1$, for each $k \in \mathbb{N}$,

$$\begin{aligned} k^{-s}v_k^T &= (k^{-s}v_k)^T(k^{-s})^{1-T} \\ &\leq \left(\left[(k^{-s}v_k)^T\right]^{1/T}\right)^T\left(\left[(k^{-s})^{1-T}\right]^{\frac{1}{1-T}}\right)^{1-T} \\ &= \left(k^{-s}v_k\right)^T, \end{aligned}$$

by Hölder's inequality and thus

$$k^{-s}w_k^{T_k} \leq k^{-s}w_k + \left(\ k^{-s}v_k\right)^T.$$

Hence $X = (X_k) \in c^F\,[M,\ \Delta,\ p,\ s]$.

References

[1] M. Başarır and M. Mursaleen, Some difference sequence spaces of fuzzy numbers, J. Fuzzy Math., **11**(2003), No. 3, 1-7.

[2] P. Diamond and P. Kloeden, Metric spaces of fuzzy sets, Theory and Applications, World Scientific, Singapore, 1994.

[3] A. Esi, On some classes of generalized difference sequences of fuzzy numbers defined by Orlicz functions, Glob. J. Pure Appl. Math., **4**(2008), No. 1, 53-64.

[4] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., **10**(1971), 379-390.

[5] M. Matloka, Sequences of fuzzy numbers, Busefal, **28**(1986), 28-37.

[6] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems, **33**(1989), 123-126.

[7] L. A. Zadeh, Fuzzy sets, Information and Control, **8**(1965), 338-353.

Scientia Magna
Vol. 7 (2011), No. 4, 92-95

The Hadwiger-Finsler reverse in an acute triangle

Marius Dragan† and Mihály Bencze‡

† Bdul Timisoara nr 35, Bl. OD6, Sc. E, Ap. 176,
061311, Bucuresti , Romania
‡ str. Harmanului nr 6, 505600, Sacele-Négyfalu, jud.
Brasov, Romania

E-mail: benczemihaly@gmail.com

Abstract In the paper the authors proposed the following inequality: In any acute triangle are true the following inequality:

$$a^2+b^2+c^2 \leq 4\sqrt{3}S+\frac{2-\sqrt{3}}{3-2\sqrt{2}}\left((a-b)^2+(b-c)^2+(c-a)^2\right). \tag{1}$$

Keywords Geometric inequalities.
2010 Mathematics Subject Classification: 26D15, 51M16, 26D99.

§1. Main results

In any triangle ABC we shall denote by $a=||\overrightarrow{BC}||$, $b=||\overrightarrow{AC}||$, $c=||\overrightarrow{AB}||$, $s=\frac{a+b+c}{2}$, R the radius of circumcircle and r the radius of incircle.

In the following we shall use the next result:

Lemma 1.1. In any triangle ABC are valid the following identities:

$$a^2+b^2+c^2=2(s^2-r^2-4Rr). \tag{2}$$

$$ab+bc+ca=s^2+r^2+4Rr. \tag{3}$$

$$a^2b^2+b^2c^2+c^2a^2=(s^2+r^2+4Rr)^2-16Rrs^2. \tag{4}$$

Lemma 1.2. In any triangle ABC are valid the following identity:

$$\cos A\cos B\cos C=\frac{s^2-r^2-4Rr-4R^2}{4R^2}. \tag{5}$$

Proof. In the following we denote $x=a^2+b^2+c^2$. From the cosinus theorem it follows

that:

$$\cos A\cos B\cos C=\frac{\prod\left(b^2+c^2+a^2\right)}{8a^2b^2c^2}=\frac{\prod\left(x-2a^2\right)}{8a^2b^2c^2}$$
$$=\frac{x^3-2\sum a^2x^2+4\sum a^2b^2x-8a^2b^2c^2}{8a^2b^2c^2}$$
$$=\frac{s^2-r^2-4Rr-4R^2}{4R^2}.$$

Theorem 1.1. In any acute triangle is true the following inequality:

$$s>2R+r. \tag{6}$$

Proof. Because in any acute triangle are true the following inequality: $\prod\cos A>0$ according with the identity (5) it follows that the inequality (6) is true.

Theorem 1.2. (Blundon)

In every triangle ABC is true the following inequality:

$$\left|s^2-\left(2R^2+10Rr-r^2\right)\right|\leq 2\left(R-2r\right)\sqrt{R\left(R-2r\right)}\ . \tag{7}$$

In the following we shall use the above result.

We denote:

$$\lambda=\frac{2-\sqrt{3}}{3-2\sqrt{2}},\qquad x=\frac{R}{r}\geq 2\ .$$

By identities (2) and (3) from Lemma 1.1 it follows that inequality (1) is equivalent with the following inequality:

$$\left(\lambda-1\right)s^2+2\sqrt{3}rs+\left(1-3\lambda\right)r^2+\left(4-12\lambda\right)Rr\geq 0\ . \tag{8}$$

We shall consider two cases:

Case 1.1. $2<x\leq\sqrt{2}+1$.

According with the right side of the Blundon's inequality in order to prove the inequality (8) it will be sufficient that:

$$\left(\lambda-1\right)\left(2R^2+10Rr-r^2-2\sqrt{R\left(R-2r\right)^3}\right)$$
$$+2\sqrt{3}r\sqrt{2R^2+10Rr-r^2-2\sqrt{R\left(R-2r\right)^3}}$$
$$+\left(1-3\lambda\right)r^2+\left(4-12\lambda\right)Rr\geq 0,$$

or in an equivalent from

$$\left(2\lambda-2\right)x^2+\left(10\lambda-10\right)x-\lambda x+1-2\left(\lambda-1\right)\sqrt{x(x-2)^3}$$
$$+2\sqrt{3}\sqrt{2x^2+10x-1-2\sqrt{x\left(x-2\right)^3}}+\left(4-12\lambda\right)x+1-3\lambda\geq 0. \tag{9}$$

Inequality (9) may be written as:

$$\left(\lambda-1\right)x^2-\left(\lambda+3\right)x+1-2\lambda+\sqrt{6x^2+30x-3-6\sqrt{x(x-2)^3}}-\left(\lambda-1\right)\sqrt{x(x-2)^3}\geq 0,$$

Or in an equivalent form:

$$(x-2)\left((\lambda-1)x+\lambda-5\right)+\sqrt{6x^2+30x-3-6\sqrt{x(x-2)^3}}-9-(\lambda-1)\sqrt{x(x-2)^3}\geq 0,$$

It follows that:

$$(x-2)\left((\lambda-1)x+\lambda-5\right)+\frac{6x^2+30x-3-6\sqrt{x\left(x-2\right)^3}}{\sqrt{6x^2+30x-3-6\sqrt{x\left(x-2\right)^3}}+9}-(\lambda-1)\sqrt{x(x-2)^3}\geq 0.$$

As $x\geq 2$ it will be sufficient to prove that:

$$(\lambda-1)x+\lambda-5+\frac{6x+42-6\sqrt{x\left(x-2\right)}}{\sqrt{6x^2+30x-3-6\sqrt{x\left(x-2\right)^3}}+9}-(\lambda-1)\sqrt{x(x-2)}>0.$$

After performing some calculation we shall obtain the inequality:

$$\left(x-\sqrt{x\left(x-2\right)}\right)\left(\lambda-1+\frac{6}{\sqrt{6x^2+30x-3-6\sqrt{x\left(x-2\right)^3}}+9}\right)$$
$$+\lambda-5+\frac{42}{\sqrt{6x^2+30x-3-6\sqrt{x\left(x-2\right)^3}}+9}+\lambda-5>0\ . \tag{10}$$

We shall consider the following function: $f,\ g,\ F:(2,\sqrt{2}+1]\to R$,

$$f\left(x\right)=x-\sqrt{x\left(x-2\right)},g\left(x\right)=6x^2+30x-3-6\sqrt{x(x-2)^3},$$

$$F\left(x\right)=f\left(x\right)\left(\lambda-1+\frac{6}{\sqrt{g(x)}+9}\right)+\frac{42}{\sqrt{g(x)}+9}+\lambda-5\ .$$

The inequality (10) may be written in an equivalent form as : $F\left(x\right)>0,\forall x\in(2,\ \sqrt{2}+1]$.

In the following we shall prove that F is a decreasing function. It will be sufficient to prove that f is an decreasing function as : $f'\left(x\right)=1-\frac{x-1}{\sqrt{x^2-2x}}<0$ it follows that f is a decresing function.

Also :

$$g'\left(x\right)=12x+30-\frac{3}{\sqrt{x\left(x-2\right)^3}}\left(\left(x-2\right)^3+3x\left(x-2\right)^2\right)$$
$$=\frac{6\left(\left(2x+5\right)\sqrt{x\left(x-2\right)}-\left(x-2\right)\left(2x-1\right)\right)}{\sqrt{x\left(x-2\right)}}$$
$$=\frac{6\left(\sqrt{x\left(2x+5\right)^2}-\sqrt{\left(x-2\right)\left(2x-1\right)^2}\right)}{\sqrt{x}}\ .$$

In order to prove that $g'(x) > 0$ will sufficient to prove that: $x(2x+5)^2 > (x-2)(2x-1)^2$ which is equivalent with the inequality $32x^2 + 16x + 2 > 0$. It follows that F is a decreasing function on the $(2,\ \sqrt{2}+1]$ interval. We shall obtain $F(x) > F(\sqrt{2}+1)$.

As :

$$F\left(\sqrt{2}+1\right) = (\lambda-1)\left(\sqrt{2}+1\right) - 4 + \frac{6\sqrt{2}+42}{\sqrt{51+36\sqrt{2}}+9} = 0.$$

We shall obtain $F(x) > 0,\ \forall\ x \in (2,\ \sqrt{2}+1]$.

Case 1.2. $x > \sqrt{2}+1$.

In order to prove the inequality (8) according theorem (1) will be sufficient to prove that : $(\lambda-1)\left(4x^2+4x+1\right) + 2\sqrt{3}(2x+1) + (4-12\lambda) + 1 - 3\lambda > 0$ or in an equivalent form:

$$\lambda\left(2x^2-4x-1\right) > 2x^2 - 2\sqrt{3}x - \sqrt{3}\ . \tag{11}$$

As $x > \sqrt{2}+1 > \frac{\sqrt{6}}{2}+1$ it follows that : $2x^2-4x-1 > 0$.

In order to prove the inequality (11) it will be sufficient to prove that :

$$\lambda > \frac{2x^2-2\sqrt{3}x-\sqrt{3}}{2x^2-4x-1}. \tag{12}$$

We shall consider the function: $f : (\sqrt{2}+1,\ +\infty) \to R$

$$f(x) = \frac{2x^2-2\sqrt{3}x-\sqrt{3}}{2x^2-4x-1}.$$

We have:

$$f'(x) = \frac{\left(2\sqrt{3}-4\right)x^2 + \left(2\sqrt{3}-2\right)x - \sqrt{3}}{\left(2x^2-4x+1\right)^2}.$$

As: $\triangle = 40 - 24\sqrt{3} > 0$ it follows that : $f'(x) < 0\,(\forall)\, x \in \left(\sqrt{2}+1,\ +\infty\right)$.

We shall obtain: $f(x) < f\left(\sqrt{2}+1\right) = \lambda$ and we have the inequality of the statement.

References

[1] Cezar Lupu, Constantin Mateescu, Vlad Matei and Mihai Opincaru , A refinement of the Finsler-Hodwinger reverse inequality , GMA. 2010, No. 1-2.

[2] D. S. Mitrinovic, J. E. Pekaric and V. Volonec, Recent Advances in Geometric inequalities.

[3] Sorin Radulescu, Marius Dragan and I. V. Maftei, Some consequences of W. J. Blundons inequalitiy, GMB. 2011, No. 1.

[4] OctogonMathematical Magazine, 1993-2011.

Scientia Magna
Vol. 7 (2011), No. 4, 96-103

On mean value computation of a number theoretic function[1]

Qianli Yang† and Chunlan Fu‡

† Department of Mathematics, Weinan Teacher's University,
Weinan, Shaanxi, 714000, P. R. China
‡ School of Sciences, South China University of Technology,
Guangzhou, Guangdong, 510640, P. R. China

Abstract The power function of p in the factorization of $n!$. It's mean value calculation were studied, methods of the elementary number theory and analytic theory and average method were adopted, precise calculation formulas of once mean value of function $b(n,\ p)$ and second mean value of function $b(n,\ p)$ were got, this conclusion for studying integer $n!$ standard decomposition type and relevant property has important impetus.

Keywords Integer standard decomposition type, number theoretic function, mean value, co -mputing formula.

§1. Introduction and results

In problem 68 of [1], Professor F. Smarandache asked us to study the properties of the sequences $k = e_p(n)$, p is a prime, k denotes the power of p in the factorization of n [2]. Studies the number sequences in the integral expression [3] and [5]. Studies the mean computing problems of digital sum. In this paper, we studied the power function of p in the factorization of $n!$ and it's mean value calculation, let p is a prime, $\alpha(n,\ p)$ denotes the power of p in the factorization of $n!$. We give an exact computing formula of the mean value $\sum\limits_{n<N}\alpha(n,\ p)$ and completes the proof. First we give two definitions as follows:

Definition 1.1. Let p $(p \geqslant 2)$ be a fixed prime, n be any positive integer and

$$n = a_1 p + a_2 p^2 + \cdots + a_s p^s,\ (0 \leqslant a_i < p,\ i = 1,\ 2, \cdots, s).$$

Let $a(n,\ p) = a_1 + a_2 + \cdots + a_s$ as function of integer sum in base p; Let $A_k(N,\ p) = \sum\limits_{n<N} a^k(n,\ p)(k = 1,\ 2)$ as once mean value of function $a(n,\ p)$ and second mean value of function $a(n,\ p)$. Set $B\,(N,\ p) = \sum\limits_{n<N} na\,(n,\ p)$ as once mean value of function $na\,(n,\ p)$.

Definition 1.2. Let p $(p \geqslant 2)$ be a fixed prime, n be any positive integer and

$$n! = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t},$$

[1]This paper is supported by the N. S. F. of project Shanxi Province (2010JM1009) and E. S. F. of project of Shanxi Province (2010JK540).

where p_i is a prime. Call $b(n,\ p)$ as the power function of p in the factorization of $n!$; Call $C_k(N,\ p) = \sum\limits_{n<N} b^k(n,\ p)(k = 1,\ 2)$ as $k-$th mean value of function $b(n,\ p)$ and second mean value of function $b(n,\ p)$.

In this paper $[x]$ denotes the greatest integer not exceeding x.

Theorem 1.1. Let $p(p \geq 2)$ be a prime, N be a fixed positive integer, assume N as follows, $N = a_1p + a_2p^2 + \cdots + a_sp^s,\ (0 \leqslant a_i < p,\ i = 1,\ 2,\ \cdots,\ s)$, then

$$\begin{aligned} C(N,\ p) &= \sum_{n<N} b(n,\ p) \\ &= \frac{1}{P-1}\left(\frac{N(N-1)}{2} - \left(\sum_{i=1}^{s}\frac{(p-1)i}{2} + \sum_{j=1}^{i} a_j - \frac{a_i+1}{2}\right)a_ip^i\right) \end{aligned}$$

or

$$\begin{aligned} C(N,\ p) &= \frac{1}{p-1}\left(\frac{N(N-1)}{2} - \sum_{i=1}^{[\log_pN]}\left(\frac{(p-1)i}{2} + \sum_{j=1}^{i}\left(\left[\frac{N}{p^j}\right] - \left[\frac{N}{p^{j+1}}\right]p\right)\right.\right. \\ &\quad \left.\left. - \frac{\left(\left[\frac{N}{p^i}\right] - \left[\frac{N}{p^{i+1}}\right]p\right)+1}{2}\right)\left(\left[\frac{N}{p^i}\right] - \left[\frac{N}{p^{i+1}}\right]p\right)p^i\right). \end{aligned}$$

Theorem 1.2.

$$\begin{aligned} C_2(N,\ p) &= \frac{1}{(p-1)^2}\left\{\varphi_2(N) + \sum_{i=1}^{s} p^{i-1}\left(a_i(i-2)\varphi_2(p) + \left(1-2p^i\right)p\varphi_2(a_i) + \right.\right. \\ &\quad (p-1)\varphi_1(i)\varphi_1(p) + 2i\left(1-p^i\right)\varphi_1(a_i)\varphi_1(p) + 2\left(a_ii\sum_{j=1}^{i-1}a_j + i - 1\right)\varphi_1(p) \\ &\quad + p\varphi_1(a_i)\sum_{j=1}^{i-1}a_j + pa_i\left(\sum_{j=1}^{i-1}a_j\right)^2 - a_i\frac{p^{i-2}-1}{2}p^2 - 2a_i\frac{{\varphi_1}^2(p)}{p}\sum_{j=1}^{i-1}jp^j\Bigg) \\ &\quad -2\sum_{i=1}^{s-1}a_i\left(\varphi_1\left(\sum_{j=i+1}^{s}a_jp^j\right) + a_ip^i\sum_{j=i+1}^{s}a_jp^j + p^i\left(\sum_{j=i+1}^{s}\left(\frac{p-1}{2}j\right.\right.\right. \\ &\quad \left.\left.\left. + \sum_{k=1}^{j}a_k - \frac{a_j+1}{2}\right)a_jp^j\right)\right)\Bigg\}. \end{aligned}$$

Corollary 1.1. Let $N = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}$, where $k_1 > k_2 > \cdots > k_s$; k_i is a positive integer, $i = 1,\ 2,\ \cdots,\ s$, then

$$C(N,\ 2) = \sum_{n<N} b(n,\ 2) = \frac{N(N-1)}{2} + \sum_{i=1}^{s}\left(k_i2^{k_i-1} + (i-1)2^{k_i}\right)$$

or

$$C(N,\ 2) = \sum_{n<N} b(n,\ 2) = \frac{N(N-1)}{2} + i - 1 + \frac{1}{2}\sum_{i=1}^{[\log_2N]}\left(k_i2^{k_i-1} + (i-1)2^{k_i}\right).$$

§2. Some lemmas and proofs

Lemma 2.1.[4] If n is expressed as follow in the base p, $n = a_0 + a_1 p + \cdots + a_k p^k$ where $0 \leqslant a_i < p$, a_i is a positive integer, $0 \leqslant i \leqslant k$ We have the exact formula,

$$a_i = \left[\frac{n}{p^i}\right] - \left[\frac{n}{p^{i+1}}\right] p, \quad k = \left[\log_p n\right]. \tag{1}$$

Lemma 2.2.[4] Let n be any positive integer, p be a prime, then

$$b(n,\ p) \equiv \sum_{i=}^{\infty} \left[\frac{n}{p^i}\right] = \frac{1}{p-1}\left(n - a(n,\ p)\right). \tag{2}$$

Lemma 2.3.[3] Let $N = a_1 p + a_2 p^2 + \cdots + a_s p^s$ where $(0 \leqslant a_i < p,\ i = 1,\ 2,\ \cdots,\ s)$. Then

$$A_1(N,\ p) = \sum_{n<N} a(n,\ p) = \sum_{i=1}^{s} \left(\frac{p-1}{2} i + \sum_{j=1}^{i} a_j - \frac{a_i + 1}{2}\right) a_i p^i \tag{3}$$

or

$$\begin{aligned} & A_1(N,\ p) \\ = & \sum_{i=1}^{s} \left(\frac{p-1}{2} i + \sum_{j=1}^{i} \left(\left[\frac{N}{p^j}\right] - \left[\frac{N}{p^{j+1}}\right] p\right) - \frac{\left(\left[\frac{N}{p^i}\right] - \left[\frac{N}{p^{i+1}}\right] p\right) + 1}{2}\right) \left(\left[\frac{N}{p^i}\right] - \left[\frac{N}{p^{i+1}}\right] p\right) p^i. \end{aligned} \tag{4}$$

Lemma 2.4.[5]

$$\begin{aligned} A_2(N,\ p) = & \sum_{i=1}^{s} \{a_i i \varphi_2(p) + p\varphi_2(a_i) + (p-1)\varphi_1(i)\varphi_1(p) + 2i\varphi_1(a_i)\varphi_1(p) \\ & + 2\left[a_i i \varphi_1(p) + p\varphi_1(a_i)\right] \sum_{j=1}^{i-1} a_j + p a_i \left(\sum_{j=1}^{i-1} a_j\right)^2 \Bigg\} p^{i-1}. \end{aligned} \tag{5}$$

Lemma 2.5.

$$\begin{aligned} B(p^k,\ p) = & \frac{p^{k-2}-1}{4} p^{k+1} + \Big(\varphi_2 p - (k-1)\varphi_1(p)\Big) p^{k-1} + {\varphi_1}^2(p) p^{k-2} \sum_{i=1}^{k-1} i p^i \\ & + \varphi_2(p) \frac{1 - p^{k-2}}{1 - p^2} p^2. \end{aligned} \tag{6}$$

Proof.

$$\begin{aligned} B\left(p^k,\ p\right) = & \sum_{n<p^k} n a\left(n,\ p\right) = \sum_{n<p^{k-1}} n a\left(n,\ p\right) + \sum_{p^{k-1} \leqslant n < 2p^{k-1}} n a\left(n,\ p\right) \\ & + \sum_{2p^{k-1} \leqslant n < 3p^{k-1}} n a\left(n,\ p\right) + \cdots + \sum_{(p-1)p^{k-1} \leqslant n < p^k} n a\left(n,\ p\right). \end{aligned}$$

Because

$$\sum_{ip^{k-1} \leqslant n < (i+1)p^{k-1}} n a\left(n,\ p\right)$$

$$\begin{aligned}
&= \sum_{0\leqslant n<p^{k-1}} \left(n+ip^{k-1}\right) a\left(n+ip^{k-1},\ p\right) \\
&= \sum_{0\leqslant n<p^{k-1}} \left(n+ip^{k-1}\right) \left(a\left(n,\ p\right)+i\right) \\
&= \sum_{0\leqslant n<p^{k-1}} na\left(n,\ p\right) + i\sum_{n<p^{k-1}} n + ip^{k-1}\sum_{n<p^{k-1}} a\left(n,\ p\right) + i^2p^{2(k-1)}.
\end{aligned}$$

From above formula

$$\begin{aligned}
B\left(p^k,\ p\right) &= pB\left(p^{k-1},\ p\right) + \frac{p\left(p-1\right)}{2}\sum_{n<p^{k-1}} n \\
&\quad + \frac{p^k\left(p-1\right)}{2}\sum_{n<p^{k-1}} a\left(n,\ p\right) + \frac{p\left(p-1\right)\left(2p-1\right)}{6}p^{2(k-1)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
pB\left(p^{k-1},\ p\right) &= p^2B\left(p^{k-2},\ p\right) + \frac{p^2\left(p-1\right)}{2}\sum_{n<p^{k-2}} n \\
&\quad + \frac{p^k\left(p-1\right)}{2}\sum_{n<p^{k-2}} a\left(n,\ p\right) + \frac{p\left(p-1\right)\left(2p-1\right)}{6}p^{2(k-2)}, \\
&\cdots\cdots
\end{aligned}$$

$$\begin{aligned}
p^{k-2}B\left(p^2,\ p\right) &= p^{k-1}B\left(p,\ p\right) + \frac{p^{k-1}\left(p-1\right)}{2}\sum_{n<p} n + \frac{p^k\left(p-1\right)}{2}\sum_{n<p} a\left(n,\ p\right) \\
&\quad + \frac{p\left(p-1\right)\left(2p-1\right)}{6}p^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
B\left(p^k,\ p\right) &= p^{k-1}B\left(p,\ p\right) + \frac{p-1}{2}\left(p\sum_{n<p^{k-1}} n + p^2\sum_{n<p^{k-2}} n + \cdots + p^{k-1}\sum_{n<p} n\right) \\
&\quad + \frac{p-1}{2}p^k\left(A\left(p^{k-1},\ p\right) + A\left(p^{k-2},\ p\right) + \cdots + A\left(p,\ p\right)\right) \\
&\quad + \frac{p\left(p-1\right)\left(2p-1\right)}{6}\left(p^{2(k-1)} + p^{2(k-2)} + \cdots + p^2\right). \qquad (7)
\end{aligned}$$

Because

$$B\left(p,\ p\right) = 1^2 + 2^2 + 3^2 + \cdots + \left(p-1\right)^2 = \frac{p\left(p-1\right)\left(2p-1\right)}{6}. \qquad (8)$$

Put (3) and (8) in (7),

$$\begin{aligned}
B\left(p^k,\ p\right) &= \frac{p^k\left(p-1\right)\left(2p-1\right)}{6} + \frac{p-1}{2}\left(\frac{p^{k+1}-p^{2k-1}}{2\left(1-p\right)} - \frac{\left(k-1\right)p^k}{2}\right) + \left(\frac{p-1}{2}\right)^2 p^k\sum_{i=1}^{k-1} ip^i \\
&\quad + \frac{p\left(p-1\right)\left(2p-1\right)}{6}\frac{p^2-p^k}{1-p^2}
\end{aligned}$$

$$= \frac{p^{k-2}-1}{4}p^{k+1} + \left(\varphi_2(p) - (k-1)\varphi_1(p)\right)p^{k-1} + {\varphi_1}^2(p)\,p^{k-2}\sum_{i=1}^{k-1} ip^i$$
$$+\varphi_2(p)\,\frac{1-p^{k-2}}{1-p^2}p^2.$$

Lemma 2.6. Let $N = a_1p + a_2p^2 + a_3p^3 + \cdots + a_sp^s$

$$\begin{aligned} B(N,\ p) &= \sum_{i=1}^{s} p^{i-1}\left\{\varphi_2(a_i)\,p^{i+1} + \varphi_1(a_i)\,\varphi_1(p)\,ip^i + a_i\frac{p^{i-2}-1}{4}p^2\right. \\ &\quad \left. + \left[\varphi_2(p) - (i-1)\,\varphi_1(p)\right]a_ip + a_i{\varphi_1}^2(p)\,p^{-1}\sum_{j=1}^{i-1} jp^j\right\} \\ &\quad + \sum_{i=1}^{s-1} a_i\left\{\varphi_1\left(\sum_{j=i+1}^{s} a_jp^j\right) + a_ip^i\sum_{j=i+1}^{s} a_jp^j\right. \\ &\quad \left. + p^i\left(\sum_{j=i+1}^{s}\left(\frac{p-1}{2}j + \sum_{k=1}^{j} a_k - \frac{a_j+1}{2}\right)a_jp^j\right)\right\}. \end{aligned}$$

Proof.

$$\begin{aligned} B(N,\ p) &= \sum_{n<N} na(n,\ p) = \sum_{n<a_1p} na(n,\ p) + \sum_{a_1p\leqslant n<N} na(n,\ p) \\ &= B(a_1p,\ p) + \sum_{0\leqslant n<N-a_1p}(n+a_1p)\,a(n+a_1p,\ p) \\ &= B(a_1p,\ p) + \sum_{n<N-a_1p}\left(na(n,\ p) + na_1 + a_1pa(n,\ p) + {a_1}^2p\right) \\ &= B(a_1p,\ p) + B(N-a_1p,\ p) + a_1\frac{(N-a_1p)(N-a_1p-1)}{2} \\ &\quad + a_1pA_1(N-a_1p,\ p) + {a_1}^2p(N-a_1p). \end{aligned}$$

Similarly

$$\begin{aligned} &B(N-a_1p,\ p) \\ &= \sum_{n<a_2p^2} na(n,\ p) + \sum_{a_2p^2\leqslant n<N-a_1p} na(n,\ p) \\ &= B\left(a_2p^2,\ p\right) + \sum_{0\leqslant n<N-a_1p-a_2p^2}\left(n+a_2p^2\right)a\left(n+a_2p^2,\ p\right) \\ &= B\left(a_2p^2,\ p\right) + \sum_{n<N-a_1p-a_2p^2}\left(na(n,\ p) + na_2 + a_2p^2a(n,\ p) + {a_2}^2p^2\right) \\ &= B\left(a_2p^2,\ p\right) + B\left(N-a_1p-a_2p^2,\ p\right) + a_2\frac{\left(N-a_1p-a_2p^2\right)\left(N-a_1p-a_2p^2-1\right)}{2} \\ &\quad + a_2p^2A_1\left(N-a_1p-a_2p^2,\ p\right) + {a_1}^2p^2\left(N-a_1p-a_2p^2\right). \end{aligned}$$

$$B\left(N-a_1p-\cdots-a_{s-2}p^{s-2},\ p\right) = B\left(a_{s-1}p^{s-1},\ p\right) + B\left(a_sp^s,\ p\right) + a_{s-1}\frac{a_sp^s\left(a_sp^s-1\right)}{2}$$

$$+a_{s-1}p^{s-1}A_1\left(a_sp^s,\ p\right)+{a_{s-1}}^2p^{s-1}\left(a_sp^s\right).$$

Therefore

$$\begin{aligned}
&B\left(N,\ p\right)\\
=\ &\sum_{i=1}^{s}B\left(a_ip^i,\ p\right)+\sum_{i=1}^{s-1}a_i\frac{\left(N-\sum\limits_{j=1}^{i}a_jp^j\right)\left(N-\sum\limits_{j=1}^{i}a_jp^j-1\right)}{2}\\
&+\sum_{i=1}^{s-1}a_ip^iA\left(N-\sum_{j=1}^{i}a_jp^j,\ p\right)+\sum_{i=1}^{s-1}{a_i}^2p^i\left(N-\sum_{j=1}^{i}a_jp^j\right)\\
=\ &\sum_{i=1}^{s}\left(a_iB\left(p^i,\ p\right)+\frac{a_i\left(a_i-1\right)}{2}\frac{p^i\left(p^i-1\right)}{2}+\frac{a_i\left(a_i-1\right)}{2}\frac{p-1}{2}ip^{2i}\right.\\
&\left.+\frac{a_i\left(a_i-1\right)\left(2a_i-1\right)}{6}p^{2i}\right)+\sum_{i=1}^{s-1}a_ip^i\left(\sum_{j=i+1}^{s}\left(\frac{p-1}{2}j+\sum_{k=1}^{j}a_k-\frac{a_j+1}{2}\right)a_jp^j\right)\\
=\ &\sum_{i=1}^{s}\left\{\varphi_2\left(a_i\right)p^{2i}+\varphi_1\left(a_i\right)\varphi_1\left(p\right)ip^{2i-1}+a_i\frac{p^{i-2}-1}{4}p^{i+1}\right.\\
&\left.+\left(\varphi_2\left(p\right)-\left(i-1\right)\varphi_1\left(p\right)\right)a_ip^{i-1}+a_i{\varphi_1}^2\left(p\right)p^{i-2}\sum_{j=1}^{i-1}jp^j\right\}+\sum_{i=1}^{s-1}a_i\left\{\varphi_1\left(\sum_{j=i+1}^{s}a_jp^j\right)\right.\\
&\left.+a_ip^i\sum_{j=i+1}^{s}a_jp^j+p^i\left(\sum_{j=i+1}^{s}\left(\frac{p-1}{2}j+\sum_{k=1}^{j}a_k-\frac{a_j+1}{2}\right)a_jp^j\right)\right\}\\
=\ &\sum_{i=1}^{s}p^{i-1}\left\{\varphi_2\left(a_i\right)p^{i+1}+\varphi_1\left(a_i\right)\varphi_1\left(p\right)ip^i+a_i\frac{p^{i-2}-1}{4}p^2\right.\\
&\left.+\left[\varphi_2\left(p\right)-\left(i-1\right)\varphi_1\left(p\right)\right]a_ip+a_i{\varphi_1}^2\left(p\right)p^{-1}\sum_{j=1}^{i-1}jp^j\right\}\\
&+\sum_{i=1}^{s-1}a_i\left\{\varphi_1\left(\sum_{j=i+1}^{s}a_jp^j\right)+a_ip^i\sum_{j=i+1}^{s}a_jp^j\right.\\
&\left.+p^i\left(\sum_{j=i+1}^{s}\left(\frac{p-1}{2}j+\sum_{k=1}^{j}a_k-\frac{a_j+1}{2}\right)a_jp^j\right)\right\}.
\end{aligned}$$

This completes the proofs of the lemma.

§3. Proofs of the theorem

In this section, we shall complete the proof of the theorem.

Proof of Theorem 1.1. From (2) we have

$$B(N,\ P)=\sum_{n<N}b(n,\ p)\quad=\quad\sum_{n<N}\frac{1}{P-1}\left(n-a(n,\ p)\right)$$

$$= \frac{1}{p-1}\left(\frac{N(N-1)}{2} - \sum_{n<N} a(n,\ p)\right)$$
$$= \frac{1}{p-1}\left(\frac{N(N-1)}{2} - A(N,\ p)\right). \tag{9}$$

Combining (3) and (9), we have

$$B(N,\ p) = \sum_{n<N} b(n,\ p) = \frac{1}{P-1}\left(\frac{N(N-1)}{2} - \sum_{i=1}^{s}\left(\frac{(p-1)i}{2} + \sum_{j=1}^{i} a_j - \frac{a_i+1}{2}\right)a_i p^i\right).$$

Combining (1)、(4) and (9), we have

$$\begin{aligned} C(N,\ p) &= \frac{1}{p-1}\left(\frac{N(N-1)}{2} - \sum_{i=1}^{[\log_p N]}\left(\frac{(p-1)i}{2} + \sum_{j=1}^{i}\left(\left[\frac{N}{p^j}\right] - \left[\frac{N}{p^{j+1}}\right]p\right)\right.\right. \\ &\quad \left.\left. - \frac{\left(\left[\frac{N}{p^i}\right] - \left[\frac{N}{p^{i+1}}\right]p\right)+1}{2}\right)\left(\left[\frac{N}{p^i}\right] - \left[\frac{N}{p^{i+1}}\right]p\right)p^i\right). \end{aligned}$$

This completes the proof of the Theorem 1.1.

Proof of Theorem 1.2. From Lemma 2.2, we have

$$\begin{aligned} C_2(N,\ p) &= \sum_{n<N} \alpha^2(n,\ p) = \sum_{n<N} \frac{1}{(p-1)^2}(n - a(n,\ p))^2 \\ &= \frac{1}{(p-1)^2}\left(\varphi_2(N) - 2\sum_{n<N} n\alpha(n,\ p) + \sum_{n<N} a^2(n,\ p)\right). \end{aligned} \tag{10}$$

Put (2) and Lemma 2.6 in (10)

$$\begin{aligned} C_2(N,\ p) &= \frac{1}{(p-1)^2}\left\{\varphi_2(N) + \sum_{i=1}^{s} p^{i-1}\left(a_i(i-2)\varphi_2(p) + \left(1-2p^i\right)p\varphi_2(a_i) + \right.\right. \\ &\quad + (p-1)\varphi_1(i)\varphi_1(p) + 2i\left(1-p^i\right)\varphi_1(a_i)\varphi_1(p) - 2\left(a_i i\sum_{j=1}^{i-1} a_j + i - 1\right)\varphi_1(p) \\ &\quad + p\varphi_1(a_i)\sum_{j=1}^{i-1} a_j + pa_i\left(\sum_{j=1}^{i-1} a_j\right)^2 - a_i\frac{p^{i-2}-1}{2}p^2 - 2a_i\frac{{\varphi_1}^2(p)}{p}\sum_{j=1}^{i-1} jp^j\Bigg) \\ &\quad - 2\sum_{i=1}^{s-1} a_i\left(\varphi_1\left(\sum_{j=i+1}^{s} a_j p^j\right) + a_i p^i\sum_{j=i+1}^{s} a_j p^j\right. \\ &\quad \left.\left. + p^i\left(\sum_{j=i+1}^{s}\left(\frac{p-1}{2}j + \sum_{k=1}^{j} a_k - \frac{a_j+1}{2}\right)a_j p^j\right)\right)\right\}. \end{aligned}$$

This completes the proof of the Theorem 1.2.

Acknowledgments

The authors express many thanks to the anonymous referees for valuable suggestions and comments.

References

[1] F. Smarandache, only problem, not solutions, Xiquan Publ. House. Chicage, 1993, 54-55.

[2] Li Hailong and M. Toda, Elaboration of some results of Srivastava and Choi, Journal of Analysis and Application, **25**(2006), 517-533.

[3] Li Hailong, On base n and related counting function, Pure and Applied Mathematics, **18**(2002), No. 1, 13-15.

[4] Li Hailong, Exponential function of integer factorization and its mean value calculation, Pure and applied mathematics, **19**(2003), No. 1, 219-222.

[5] Li Hailong, Base n Nonzero Digital Multiplication Function and Its Mean Value Formula, Mathematics of practice and cognition, **22**(2002), No. 4, 683-686.

Scientia Magna
Vol. 7 (2011), No. 4, 104-108

On Kummer's fourier series for $\log\Gamma(x)$[1]

Yulin Lu†, Youcheng Li‡ and Hailong Li♯

† ♯ Weinan Teachers College, Weinan, Shaanxi, 714000, P. R. China
‡ Weinan Vocational Technical College, Weinan, Shaanxi, 714000, P. R. China

Abstract In Espinosa and Moll [1], the integral $\int_0^1 \log\Gamma(x)\zeta(z,\ x)dx$ is evaluated using the Fourier coefficients a_n, b_n of $\log\Gamma(x)$ found in [2]. However, in both [2] and [5], these coefficients appear as $\frac{1}{2}a_n$ and $\frac{1}{2}b_n$. In view of this, we derive in this note Kummer's Fourier series for $\log\Gamma(x)$ by which we evaluate the above integral.

Keywords Gamma function, kummer's fourier series, malmsten's formula, hurwitz zeta-fun -ction, class function.

§1. Introduction and the main result

Let $\zeta(s,\ x),\ (0 < x \leqslant 1)$ denote the Hurwitz zeta-function defined by

$$\zeta(s,\ x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}\ ,\ \sigma = \mathrm{Re}\ s > 1 \tag{1}$$

and let $\zeta(s) = \zeta(s,\ 1)$ denote the Riemann zeta-function. Let $\Gamma(x)$ denote the gamma function defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt,\ \ \mathrm{Re}\, x > 0. \tag{2}$$

The Hurwitz zeta-function $\zeta(s,\ x)$ satisfies the functional equation $(0 < x \leqslant 1)$

$$\zeta(s,\ x) = -\frac{i\Gamma(1-z)}{(2\pi)^{1-z}}\left(e^{\frac{\pi iz}{2}}l_{1-z}(x) - e^{-\frac{\pi iz}{2}}l_{1-z}(1-x)\right), \tag{3}$$

where

$$l_s(x) = \sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{n^s},\ \ \sigma > 1 \tag{4}$$

is the polylogarithm function.

We understand that these functions are meromorphically continued over C. Espinosa and Moll [1, Example 6.1] and Hashimoto [3, Corollary 3] evaluate the integral $\int_0^1 \log\Gamma(x)\zeta(z,\ x)dx$ in closed form. The former appealed to their theorem and used the Fourier coefficients a_n and b_n of $\log\Gamma(x)$ [2], while the latter used Lerch's formala

$$\log\frac{\Gamma(x)}{\sqrt{2\pi}} = \zeta'(0,\ x). \tag{5}$$

[1]This work is supported by the N. S. F. (10271093) of P. R. China.

However, both in [2] and [5], the Fourier coefficients are stated as $\frac{1}{2}a_n$ and $\frac{1}{2}b_n$ and do not lead directly to the Fourier expansion (11) below for $log\Gamma(x)$. In view of this, we shall elaborate on the proof in [5] for the same, and using it, we shall prove the following

Theorem 1.1. For $0 < \text{Re } z < 1$, we have

$$\int_0^1 \log\Gamma(x)\zeta(z,\ x)dx = (\gamma + \log 2\pi)\frac{\zeta(z-1)}{z-1} - \frac{1}{\pi}\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\cos\frac{\pi z}{2}\zeta'(2-z) + \frac{1}{2}\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{\pi z}{2}\zeta(2-z), \tag{6}$$

where γ indicates the Euler constant.

§2. Proof of theorem

We need some lemmas.

Lemma 1. For $0 < \text{Re } z < 1$ we have

$$\int_0^1 \zeta(z,\ x)dx = 0, \tag{7}$$

$$\int_0^1 x\zeta(z,\ x)dx = -\frac{\zeta(z-1)}{z-1}. \tag{8}$$

For $\text{Re } z > 0$

$$\int_0^1 A_1(x)\zeta(z,\ x)dx = -\int_0^1 \log(2\sin\pi x)\zeta(z,\ x)dx = \frac{\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{\pi z}{2}\zeta(2-z), \tag{9}$$

where $A_1(x)$ indicates the first Clausen function

$$A_1(x) = -\log(2\sin\pi x),\ 0 < x < 1. \tag{10}$$

Proof can he found in [1] or [3].

Lemma 2. (Kummer's Fourier series, [5, p. 17, (35)]). For $0 < x < 1$,

$$\log\frac{\Gamma(x)}{\sqrt{2\pi}} = \frac{1}{2}A_1(x) - (\gamma + \log 2\pi)B_1(x) + \sum_{n=1}^{\infty}\frac{\log n}{\pi n}\sin 2\pi nx, \tag{11}$$

where $B_1(x) = x - \frac{1}{2}$ is the first Bernoulli polynomial.

Proof. We reproduce the proof sketched in [5, p.17] more in detail.

By the reciprocity relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \tag{12}$$

we may write for $0 < \text{Re } z < 1$,

$$\log\Gamma(z) = -\frac{1}{2}\log\frac{\sin\pi z}{\pi} + \frac{1}{2}\left(\log\Gamma(z) - \log\Gamma(1-z)\right).$$

Substituting Malmsten's formula [5, p.16, (25)],

$$\log\Gamma(z)=\int_0^\infty\left(z-1-\frac{1-e^{-(z-1)t}}{1-e^{-t}}\right)\frac{e^t}{t}dt, \tag{13}$$

we deduce that

$$\log\Gamma(z)=\frac{1}{2}\log 2\pi+\frac{1}{2}A_1(z)+\frac{1}{2}\int_0^\infty\left((2z-1)e^{-t}+\frac{e^{-(z-1)t}-e^{-zt}}{1-e^{-t}}e^{-t}\right)\frac{dt}{t}. \tag{14}$$

Writing $\frac{e^{-(z-1)t}-e^{-zt}}{1-e^{-t}}e^{-t}$ as $\frac{\sinh\left(\frac{1}{2}-z\right)t}{\sinh\frac{t}{2}}$, we deduce Kummer's expression [5, p.16, (35)]

$$\log\frac{\Gamma(z)}{\sqrt{2\pi}}=A_1(z)+\frac{1}{2}\int_0^\infty\left((2z-1)\,e^{-t}+\frac{\sinh\left(\frac{1}{2}-z\right)t}{\sinh\frac{t}{2}}\right)\frac{dt}{t}\ ,\ 0<\text{Re }z<1. \tag{15}$$

Now we compute the Fourier sine coefficient

$$\begin{aligned}b_n&=2\int_0^1\sin 2\pi nx\frac{1}{2}\int_0^\infty\left(2xe^{-t}+\frac{\sinh\left(\frac{1}{2}-x\right)t}{\sinh\frac{t}{2}}\right)\frac{dt}{t}dx\\&=\int_0^\infty\frac{dt}{t}\left(2e^{-t}\int_0^1 x\sin 2\pi nxdx+\frac{1}{\sinh\frac{t}{2}}\int_0^1\sin 2\pi nx\sinh\left(\frac{1}{2}-x\right)tdx\right).\end{aligned}$$

Since

$$\begin{aligned}\sin 2\pi nx\sinh\left(\frac{1}{2}-x\right)t&=-i\sin 2\pi nx\sinh\left(\left(\frac{1}{2}-x\right)it\right)\\&=\frac{i}{2}\left(\cos\left((2\pi n+it)\,x-\frac{1}{2}it\right)-\cos\left((2\pi n-it)\,x+\frac{1}{2}it\right)\right).\end{aligned}$$

We obtain

$$\begin{aligned}b_n&=\int_0^\infty\left(\frac{1}{2\pi n}\frac{e^{-t}}{t}+\frac{1}{t\sinh\frac{t}{2}}\left[\frac{1}{2\pi n+it}\sin\left((2\pi n+it)\,x-\frac{1}{2}it\right)\right.\right.\\&\quad\left.\left.-\ \frac{1}{2\pi n-it}\sin\left((2\pi n-it)\,x+\frac{1}{2}it\right)\right]_0^1\right)dt\\&=\int_0^\infty\left(\frac{1}{2\pi n}\frac{e^{-t}}{t}+\frac{i}{2}\frac{1}{t\sinh\frac{t}{2}}2\sinh\frac{it}{2}\left(\frac{1}{2\pi n+it}+\frac{1}{2\pi n-it}\right)\right)dt.\end{aligned}$$

Hence

$$b_n=\int_0^\infty\left(\frac{4\pi n}{t^2+(2\pi n)^2}-\frac{1}{\pi n}e^{-t}\right)\frac{dt}{t}=\frac{1}{\pi n}\int_0^\infty\left(\frac{1}{t^2+1}-e^{-t}\right)\frac{dt}{t}. \tag{16}$$

We rewrite the integrand as $\left(\frac{1}{1+t^2}-\cos t\right)\frac{1}{t}+\frac{e^{-t}-e^{-2\pi nt}}{t}+\left(\cos t-e^{-t}\right)\frac{1}{t}$ and apply the following formulas.

$$\gamma=\int_0^\infty\left(\frac{1}{1+t^2}-\cos t\right)\frac{dt}{t} \tag{17}$$

[5, p.6, (36) = P.5, (28); in (36), t must be t^2],

$$\log\ z=\int_0^\infty\left(e^{-t}-e^{-zt}\right)\frac{dt}{t}\ ,\quad\text{Re }z>0\qquad\text{[5, p.16, (28)]} \tag{18}$$

and

$$\gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t}\right)\frac{dt}{t} \qquad [5,\ \text{p.5, (24)}]. \tag{19}$$

We need to prove that

$$\int_0^\infty \left(\cos t - e^{-t}\right)\frac{dt}{t} = 0.$$

Which can be done by subtracting (17) from (19) and noting that the resulting integral

$$\int_0^\infty \frac{t^2-t}{(t^2+1)(t+1)}\frac{dt}{t} = 0.$$

Hence it follows that

$$b_n = \frac{1}{\pi n}(\gamma + \log 2\pi n). \tag{20}$$

From (15) and (20), we now deduce for $0 < x < 1$ that

$$\log\frac{\Gamma(x)}{\sqrt{2\pi}} = \frac{1}{2}A_1(x) + \sum_{n=1}^\infty b_n \sin 2\pi nx = \sum_{n=1}^\infty \frac{1}{2n}\cos 2\pi nx + \sum_{n=1}^\infty \frac{\gamma + \log 2\pi n}{\pi n}\sin 2\pi nx. \tag{21}$$

Where we used the Founer series for $A_1(x)$:

$$A_1(x) = -\log 2\pi nx = \sum_{n=1}^\infty \frac{1}{n}\cos 2\pi nx. \tag{22}$$

Recalling the counterpart of (22) for $B_1(x)$, we conclude (11) from (21),

$$B_1(x) = -\frac{1}{\pi}\sum_{n=1}^\infty \frac{\sin 2\pi nx}{n}\ ,\ 0 < x < 1. \tag{23}$$

This completes the proof.

Proof. Solving (11) for $A_1(x)$ and substituting in (9), we deduce that

$$\int_0^1 \left(\log\frac{\Gamma(x)}{\sqrt{2\pi}} + (\gamma + \log 2\pi)(x - \frac{1}{2}) - \sum_{n=1}^\infty \frac{\log n}{\pi n}\sin 2\pi nx\right)\zeta(z,\ x)dx$$
$$= \frac{1}{2}\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{\pi z}{2}\zeta(2-z)$$

whence we see, on using (7) and (8), that

$$\int_0^1 \log\Gamma(x)\zeta(z,\ x)dx = \int_0^1 \log\frac{\Gamma(x)}{\sqrt{2\pi}}\zeta(z,\ x)dx$$
$$= (\gamma + \log 2\pi)\frac{\zeta(z-1)}{z-1} + \frac{1}{2}\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{\pi z}{2}\zeta(2-z) + I \tag{24}$$

say, where

$$I = \frac{1}{\pi}\int_0^1 \sum_{n=1}^\infty \frac{\log n}{n}\sin 2\pi nx\zeta(z,\ x)dx. \tag{25}$$

Changing the order of integration and summation in (25), we find that

$$I = \frac{1}{\pi}\sum_{n=1}^\infty \frac{\log n}{n}\int_0^1 \sin 2\pi n\zeta(z,\ x)dx.$$

The Fourier sine coefficients $b_n = 2\int_0^1 \sin 2\pi nx\zeta(z,\ x)dx$ are evaluated in [1, Proposition 2.1] but we give a proof in the lines of that of Corollary 1 [4]. We use (3) and the expression $\sin 2\pi nx = \frac{e^{2\pi inx}-e^{-2\pi inx}}{2i}$ and proceed as in [4]:

$$\begin{aligned} b_n &= -2\frac{\Gamma(1-z)}{2\,(2\pi)^{1-z}}\int_0^1 \left(e^{2\pi inx}-e^{-2\pi inx}\right)\left(e^{\frac{\pi iz}{2}}l_{1-z}(x)-e^{\frac{-\pi iz}{2}}l_{1-z}(1-x)\right)dx \\ &= -\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\int_0^1 \left(-e^{\frac{-\pi iz}{2}}e^{2\pi inx}l_{1-z}\,(1-x)-e^{\frac{\pi iz}{2}}e^{-2\pi inx}l_{1-z}\,(x)\right)dx \\ &= \frac{\Gamma(1-z)}{(2\pi)^{1-z}}n^{1-z}\cos\frac{\pi z}{2}. \end{aligned}$$

Hence

$$I = \frac{1}{\pi}\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\cos\frac{\pi z}{2}\sum_{n=1}^{\infty}\frac{\log n}{n^{2-z}} = -\frac{1}{\pi}\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\cos\frac{\pi z}{2}\zeta'(2-z). \tag{26}$$

Substituting (26) in (24), we conclude (6), completing the proof.

Acknowledgement

The author would like to thank Professors S. Kanemitsu for their help and useful discussions.

References

[1] O. Espinosa and V. Moll, On some integrals involving the Hurwitz zeta-function, Ramanujan J., **6**(2002), 259-188.

[2] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products, Academic Press, New York, 1994.

[3] M. Hashimoto, Examples of the Hurwitz transforms, to appear.

[4] S. Kanemitsu, Y. Tanigawa and J. H. Zhang, Evaluation of Spannanintegrals of the product of zeta-functions, to appear.

[5] H. M. Srivastava and J. S. Choi, Series associated with the Zeta and releted functions, Kluwer Academic Publishers, Dordrecht etc, 2001.

Scientia Magna
Vol. 7 (2011), No. 4, 109-113

On two inequalities for the composition of arithmetic functions

József Sándor

Department of Mathematics, Babeş-Bolyai University,
Str. Kogălniceanu nr. 1, 400084, Cluj-Napoca, Romania
E-mail: jsandor@math.ubbcluj.ro

Abstract Let f, g be arithmetic functions satisfying certain conditions. We prove the inequalities $f(g(n)) \leq 2n - \omega(n) \leq 2n - 1$ and $f(g(n)) \leq n + \omega(n) \leq 2n - 1$ for any $n \geq 1$, where $\omega(n)$ is the number of distinct prime factors of n. Particular cases include $f(n) =$ Smarandache function, $g(n) = \sigma(n)$ or $g(n) = \sigma^*(n)$.
Keywords Arithmetic functions, inequalities.
AMS Subject Classification: 11A25.

§1. Introduction

Let $S(n)$ be the Smarandache (or Kempner-Smarandache) function, i.e., the function that associates to each positive integer n the smallest positive integer k such that $n|k!$. Let $\sigma(n)$ denote the sum of distinct positive divisors of n, while $\sigma^*(n)$ the sum of distinct unitary divisors of n (introduced for the first time by E. Cohen, see e.g. [7] for references and many informations on this and related functions). Put $\omega(n) =$ number of distinct prime divisors of n, where $n > 1$. In paper [4] we have proved the inequality

$$S(\sigma(n)) \leq 2n - \omega(n), \tag{1}$$

for any $n > 1$, with equality if and only if $\omega(n) = 1$ and $2n - 1$ is a Mersenne prime.

In what follows we shall prove the similar inequality

$$S(\sigma^*(n)) \leq n + \omega(n) \tag{2}$$

for $n > 1$. Remark that $n + \omega(n) \leq 2n - \omega(n)$, as $2\omega(n) \leq n$ follows easily for any $n > 1$. On the other hand $2n - \omega(n) \leq 2n - 1$, so both inequalities (1) and (2) are improvements of

$$S(g(n)) \leq 2n - 1, \tag{3}$$

where $g(n) = \sigma(n)$ or $g(n) = \sigma^*(n)$.

We will consider more general inequalities, for the composite functions $f(g(n))$, where f, g are arithmetical functions satisfying certain conditions.

§2. Main results

Lemma 2.1. For any real numbers $a \geq 0$ and $p \geq 2$ one has the inequality

$$\frac{p^{a+1}-1}{p-1} \leq 2p^a - 1, \tag{4}$$

with equality only for $a = 0$ or $p = 2$.

Proof. It is easy to see that (4) is equivalent to

$$(p^a - 1)(p-2) \geq 0,$$

which is true by $p \geq 2$ and $a \geq 0$, as $p^a \geq 2^a \geq 1$ and $p - 2 \geq 0$.

Lemma 2.2. For any real numbers $y_i \geq 2$ $(1 \leq i \leq r)$ one has

$$y_1 + \cdots + y_r \leq y_1 \cdots y_r \tag{5}$$

with equality only for $r = 1$.

Proof. For $r = 2$ the inequality follows by $(y_1 - 1)(y_2 - 1) \geq 1$, which is true, as $y_1 - 1 \geq 1$, $y_2 - 1 \geq 1$. Now, relation (5) follows by mathematical induction, the induction step $y_1 \cdots y_r + y_{r+1} \leq (y_1 \cdots y_r)y_{r+1}$ being an application of the above proved inequality for the numbers $y_1' = y_1 \cdots y_r$, $y_2' = y_{r+1}$.

Now we can state the main results of this paper.

Theorem 2.1. Let $f,\ g : \mathbb{N} \to \mathbb{R}$ be two arithmetic functions satisfying the following conditions:

(i) $f(xy) \leq f(x) + f(y)$ for any $x,\ y \in \mathbb{N}$;

(ii) $f(x) \leq x$ for any $x \in \mathbb{N}$;

(iii) $g(p^\alpha) \leq 2p^\alpha - 1$, for any prime powers p^α (p prime, $\alpha \geq 1$);

(iv) g is multiplicative function.

Then one has the inequality

$$f(g(n)) \leq 2n - \omega(n),\ n > 1. \tag{6}$$

Theorem 2.2. Assume that the arithmetical functions f and g of Theorem 2.1 satisfy conditions (i), (ii), (iv) and

(iii)$'$ $g(p^\alpha) \leq p^\alpha + 1$ for any prime powers p^α.

Then one has the inequality

$$f(g(n)) \leq n + \omega(n),\ n > 1. \tag{7}$$

Proof of Theorem 2.1. As $f(x_1) \leq f(x_1)$ and $f(x_1x_2) \leq f(x_1) + f(x_2)$, it follows by mathematical induction, that for any integers $r \geq 1$ and $x_1, \cdots, x_r \geq 1$ one has

$$f(x_1 \cdots x_r) \leq f(x_1) + \cdots + f(x_r). \tag{8}$$

Let now $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1$ be the prime factorization of n, where p_i are distinct primes and $\alpha_i \geq 1$ $(i = 1, \cdots, r)$. Since g is multiplicative, by inequality (8) one has

$$f(g(n)) = f(g(p_1^{\alpha_1}) \cdots g(p_r^{\alpha_r})) \leq f(g(p_1^{\alpha_1})) + \cdots + f(g(p_r^{\alpha_r})).$$

By using conditions (ii) and (iii), we get

$$f(g(n)) \leq g(p_1^{\alpha_1}) + \cdots + g(p_r^{\alpha_r}) \leq 2(p_1^{\alpha_1} + \cdots + p_r^{\alpha_r}) - r.$$

As $p_i^{\alpha_i} \geq 2$, by Lemma 2.2 we get inequality (6), as $r = \omega(n)$.

Proof of Theorem 2.2. Use the same argument as in the proof of Theorem 2.1, by remarking that by (iii)$'$

$$f(g(n)) \leq (p_1^{\alpha+1} + \ldots + p_r^{\alpha_r}) + r \leq p_1^{\alpha_1} \ldots p_r^{\alpha_r} + r = n + \omega(n).$$

Remark 2.1. By introducing the arithmetical function $B^1(n)$ (see [7], Ch.IV.28)

$$B^1(n) = \sum_{p^\alpha \| n} p^\alpha = p_1^{\alpha_1} + \ldots + p_r^{\alpha_r}$$

(i.e., the sum of greatest prime power divisors of n), the following stronger inequalities can be stated:

$$f(g(n)) \leq 2B^1(n) - \omega(n), \tag{6$'$}$$

(in place of (6)); as well as:

$$f(g(n)) \leq B^1(n) + \omega(n), \tag{7$'$}$$

(in place of (7)).

For the average order of $B^1(n)$, as well as connected functions, see e.g. [2], [3], [8], [7].

§3. Applications

1. First we prove inequality (1).

Let $f(n) = S(n)$. Then inequalities (i), (ii) are well-known (see e.g. [1], [6], [4]). Put $g(n) = \sigma(n)$. As $\sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1}$, inequality (iii) follows by Lemma 2.1. Theorem 2.1 may be applied.

2. Inequality (2) holds true.

Let $f(n) = S(n)$, $g(n) = \sigma^*(n)$. As $\sigma^*(n)$ is a multiplicative function, with $\sigma^*(p^\alpha) = p^\alpha + 1$, inequality (iii)$'$ holds true. Thus (2) follows by Theorem 2.2.

3. Let $g(n) = \psi(n)$ be the Dedekind arithmetical function; i.e., the multiplicative function whose value of the prime power p^α is

$$\psi(p^\alpha) = p^{\alpha-1}(p+1).$$

Then $\psi(p^\alpha) \leq 2p^\alpha - 1$ since

$$p^\alpha + p^{\alpha-1} \leq 2p^\alpha - 1 \Longleftrightarrow p^{\alpha-1} + 1 \leq p^\alpha \leq p^{\alpha-1}(p-1) \geq 0,$$

which is true, with strict inequality.

Thus Theorem 2.1 may be applied for any function f satisfying (i) and (ii).

4. There are many functions satisfying inequalities (i) and (ii) of Theorems 2.1 and 2.2.

Let $f(n) = \log \sigma(n)$. As $\sigma(mn) \leq \sigma(m)\sigma(n)$ for any $m,\ n \geq 1$, relation (i) follows. The inequality $f(n) \leq n$ follows by $\sigma(n) \leq e^n$, which is a consequence of e.g. $\sigma(n) \leq n^2 < e^n$ (the last inequality may be proved e.g. by induction).

Remark 3.1. More generally, assume that $F(n)$ is a submultiplicative function, i.e., satisfying

$$F(mn) \leq F(m)F(n), \quad \text{for } m,\ n \geq 1. \tag{i$'$}$$

Assume also that

$$F(n) \leq e^n. \tag{ii$'$}$$

Then $f(n) = \log F(n)$ satisfies relations (i) and (ii).

5. Another nontrivial function, which satisfies conditions (i) and (ii) is the following

$$f(n) = \begin{cases} p, & \text{if } n = p \text{ (prime);} \\ 1, & \text{if } n = \text{composite or } n = 1. \end{cases} \tag{9}$$

Clearly, $f(n) \leq n$, with equality only if $n = 1$ or $n =$ prime. For $y = 1$ we get $f(x) \leq f(x) + 1 = f(x) + f(1)$; when $x,\ y \geq 2$ one has

$$f(xy) = 1 \leq f(x) + f(y).$$

6. Another example is

$$f(n) = \Omega(n) = \alpha_1 + \cdots + \alpha_r, \tag{10}$$

for $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$; i.e., the total number of prime factors of n. Then $f(mn) = f(m) + f(n)$, as $\Omega(mn) = \Omega(m) + \Omega(n)$ for all $m,\ n \geq 1$. The inequality $\Omega(n) < n$ follows by $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \geq 2^{\alpha_1 + \cdots + \alpha_r} > \alpha_1 + \cdots + \alpha_r$.

7. Define the **additive analogue** of the sum of divisors function σ, as follows: If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the prime factorization of n, put

$$\Sigma(n) = \Sigma\left(\frac{p^{\alpha+1} - 1}{p - 1}\right) = \sum_{i=1}^{r} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}. \tag{11}$$

As $\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1} - 1}{p - 1}$, and $\frac{p^{\alpha+1}-1}{p-1} > 2$, clearly by Lemma 2.2 one has

$$\Sigma(n) \leq \sigma(n). \tag{12}$$

Let $f(n)$ be any arithmetic function satisfying condition (ii), i.e., $f(n) \leq n$ for any $n \geq 1$. Then one has the inequality:

$$f(\Sigma(n)) \leq 2B^1(n) - \omega(n) \leq 2n - \omega(n) \leq 2n - 1, \tag{13}$$

for any $n > 1$.

Indeed, by Lemma 2.1 and Remark 2.1, the first inequality of (13) follows. Since $B^1(n) \leq n$ (by Lemma 2.2), the other inequalities of (13) will follow. An example:

$$S(\Sigma(n)) \leq 2n - 1, \tag{14}$$

which is the first and last term inequality in (13).

It is interesting to study the cases of equality in (14). As $S(m) = m$ if and only if $m = 1$, 4 or p (prime) (see e.g. [1], [6], [4]) and in Lemma 2.2 there is equality if $\omega(n) = 1$, while in Lemma 2.1, as $p = 2$, we get that n must have the form $n = 2^\alpha$. Then $\Sigma(n) = 2^{\alpha+1} - 1$ and $2^{\alpha+1} - 1 \neq 1$, $2^{\alpha+1} - 1 \neq 4$, $2^{\alpha+1} - 1 =$ prime; we get the following theorem:

There is equality in (14) iff $n = 2^\alpha$, where $2^{\alpha+1} - 1$ is a prime.

In paper [5] we called a number n almost f-perfect, if $f(n) = 2n - 1$ holds true. Thus, we have proved that n is almost $S \circ \Sigma$-perfect number, iff $n = 2^\alpha$, with $2^{\alpha+1} - 1$ a prime (where "$\circ$" denotes composition of functions).

References

[1] Ch. Ashbacher, An introduction to the Smarandache function, Erhus Univ. Press, Vail, 1995.

[2] P. Erdös and A. Ivić, Estimates for sums involving the largest prime factor of an integer and certain related additive functions, Studia Sci. Math. Hung., **15**(1980), 183-199.

[3] J. M. de Koninck and A. Ivić, Topics in arithmetical functions, Notas de Matematica, North Holland, 1980.

[4] F. Luca and J. Sándor, On the composition of a certain arithmetic function, Functiones et Approximatio, **41**(2009), 185-209.

[5] J. Sándor, On the composition of some arithmetic functions, Studia Univ. Babeş-Bolyai, Math., **34**(1989), 7-14.

[6] J. Sándor, Geometric theorems, diophantine equations and arithmetic functions, New Mexico, 2002.

[7] J. Sándor, D. S. Mitrinović and B. Crstici, Handbook of number theory I, Springer Verlag, 2006.

[8] T. Z. Xuan, On some sums of large additive number theoretic functions (Chinese), Beijing Shifan Daxue Xuebao, 1984, No. 2, 11-18.

Scientia Magna
Vol. 7 (2011), No. 4, 114-120

Compactness and proper maps in the category of generated spaces

Ghasem Mirhosseinkhani

Department of Mathematics, Hormozgan University,
P. O. Box 3995, Bandarabbas, Iran
E-mail: gh.mirhosseini@yahoo.com

Abstract In this paper we present the notion of relative $\mathcal{C}$-compactness which is a generalization of the notion of compactness in the category $Top_{\mathcal{C}}$ of $\mathcal{C}$-generated spaces. Also we shall study proper maps in $Top_{\mathcal{C}}$, in particular, we show that proper maps in the categories Top of all topological spaces, core compactly generated spaces, locally compactly generated spaces and compactly generated spaces coincide.

Keywords Generated space, compactness, proper map.
AMS Subject Classification Codes: 18B30, 54D30, 54C10.

§1. Introduction and preliminaries

Recall that a continuous map $f: X \to Y$ in the category Top of topological spaces is called proper if the product map $id_Z \times f: Z \times X \to Z \times Y$ is closed, for every space Z, where $id_Z: Z \to Z$ is the identity map. This is equivalent to f being universally closed, in the sense that every pullback of f is a closed map. There exist many characterizations of proper maps [1,3], the useful characterizations of them related to compactness are presented in [5]. In particular, a continuous map $f: X \to Y$ is proper if and only if f is a closed map and the set $f^{-1}\{y\}$ is compact for every point $y \in Y$.

Proper maps were worked categorically by Clementino, Giuli and Tholen, as the notion of c-compact maps, where c is a closure operator, see [2] for details. Also some categorically properties of proper maps are discussed in [4], [8] and [9].

Let $\mathcal{C}$ be a fixed collection of spaces, referred to as generating spaces. By a probe over a space X we mean a continuous map from one of the generating spaces to X. The $\mathcal{C}$-generated topology $\mathcal{C}X$ on a space X is the final topology of the probes over X, that is, the finest topology making all probes continuous. We say that a topological space X is $\mathcal{C}$-generated, if $X = \mathcal{C}X$. The category of continuous maps of $\mathcal{C}$-generated spaces is denoted by $Top_{\mathcal{C}}$. In particular, if $\mathcal{C}$ consists of core compact spaces, locally compact spaces, compact Hausdorff spaces, the one-point compactification of the countable discrete space respectively, then we refer to $\mathcal{C}$-generated spaces as core compactly generated spaces, locally compactly generated spaces, compactly generated spaces, sequentially generated spaces, see [7]. A collection $\mathcal{C}$ of

generating spaces is called productive if every generating space is exponentiable, and also the topological product of any two generating space is $\mathcal{C}$-generated. If $\mathcal{C}$ is productive, then $Top_{\mathcal{C}}$ is a cartesian closed category.

For $\mathcal{C}$-generated spaces X and Y, we write $X \times_{\mathcal{C}} Y$ for the binary product in the category $Top_{\mathcal{C}}$, which is equal to $\mathcal{C}(X \times Y)$, see [7]. Indeed, in general the topology of $X \times_{\mathcal{C}} Y$ is finer than the topological product $X \times Y$, but if the product space $X \times Y$ is itself $\mathcal{C}$-generated, then $X \times Y = X \times_{\mathcal{C}} Y$. A continuous map $f : X \to Y$ in the category $Top_{\mathcal{C}}$ is called proper if the product map $id_Z \times f : Z \times_{\mathcal{C}} X \to Z \times_{\mathcal{C}} Y$ is closed, for every $\mathcal{C}$-generated space Z. In this paper we shall study proper maps in $Top_{\mathcal{C}}$. We first present the notion of relative $\mathcal{C}$-compactness and $\mathcal{C}$-compactness and next we present some characterizations of those maps. All maps in this paper are assumed to be continuous.

Theorem 1.1.[7] Let $\mathcal{C}$ be a productive class of generating spaces.

(i) If X and Y are $\mathcal{C}$-generated spaces with Y exponentiable in Top, then the topological product $X \times Y$ is $\mathcal{C}$-generated.

(ii) If X and Y are $\mathcal{C}$-generated, then $X \times_{\mathcal{C}} Y = X \times_{\mathcal{E}} Y$, where $\mathcal{E}$ denotes the class of all exponentiable spaces.

(iii) If A, B, Y are $\mathcal{C}$-generated and $q : A \to B$ is a topological quotient, then the map $q \times id_Y : A \times_{\mathcal{C}} Y \to B \times_{\mathcal{C}} Y$ is a topological quotient.

§2. Relative $\mathcal{C}$-compactness and $\mathcal{C}$-compactness

A generalization of compactness is the notion of relative compactness, that is, for arbitrary subsets S and T of a topological space X, we say that S is relatively compact in T, written $S \ll T$, if, for every open cover of T, there exist finitely many elements in the cover that cover S. The notion of relative compactness, restricted to open sets, plays an important role in the study of core compact spaces (exponentiable spaces) and generated spaces [6,7]. For an arbitrary collection $\mathcal{C}$ of generating spaces, the relation $\ll_{\mathcal{C}}$, on subsets of a topological space X, is defined by $S \ll_{\mathcal{C}} T$ if and only if there is a $\mathcal{C}$-probe $P : C \to X$ and open subsets $U \ll V \subseteq C$, such that $S \subseteq P(U)$ and $P(V) \subseteq T$. If $\mathcal{C}$ is productive, then the exponential topology $[X \Rightarrow_{\mathcal{C}} Y]$ has subbasic open sets $[S, V]_{\mathcal{C}} = \{f \mid S \ll_{\mathcal{C}} f^{-1}(V)\}$, where $S \subseteq X$ is arbitrary and $V \subseteq Y$ is open [7].

In this section, we give the notion of relative $\mathcal{C}$-compactness, which is a generalization of relative compactness and $\mathcal{C}$-compactness. Also we present necessary and sufficient conditions for relative $\mathcal{C}$-compactness. A useful characterization of relative compactness is presented in [5], that is, $S \ll T$ if and only if for every space Z, every $z \in Z$ and every open subset W of $Z \times X$, such that $\{z\} \times T \subseteq W$, then $V \times S \subseteq W$, for some neighborhood V of z. This is an idea for the following definition:

Definition 2.1. Let S, T and Q be subsets of a $\mathcal{C}$-generated space X.

(i) We say that S is relatively $\mathcal{C}$-compact in T, written $S \ll^{\mathcal{C}} T$, if for every $\mathcal{C}$-generated space Z, every $z \in Z$ and every open subset W of $Z \times_{\mathcal{C}} X$, such that $\{z\} \times T \subseteq W$, then $V \times S \subseteq W$, for some neighborhood V of z.

(ii) We say that Q is $\mathcal{C}$-compact, if the relation $Q \ll^{\mathcal{C}} Q$ holds.

Remark 2.1. By definition of interior, the relation $S \ll^{\mathcal{C}} T$ holds if and only if for every $\mathcal{C}$-generated space Z and every open subset W of $Z \times_{\mathcal{C}} X$,

$$\{z \in Z \mid \{z\} \times T \subseteq W\} \subseteq \{z \in Z \mid \{z\} \times S \subseteq W\}^{\circ}.$$

Therefore a subset Q of a $\mathcal{C}$-generated space X is $\mathcal{C}$-compact if and only if for every $\mathcal{C}$-generated space Z and every open subset W of $Z \times_{\mathcal{C}} X$, the set $\{z \in Z \mid \{z\} \times Q \subseteq W\}$ is open.

Proposition 2.1. Let S and T be subsets of a $\mathcal{C}$-generated space X.

(i) If $\mathcal{C}$ is productive, then $S \ll_{\mathcal{C}} T$ implies $S \ll^{\mathcal{C}} T$.

(ii) If $\mathcal{C}$ is productive and X is a core compact space, then $S \ll T$ implies $S \ll^{\mathcal{C}} T$.

Proof. (i) Suppose $S \ll_{\mathcal{C}} T$. Then there is a $\mathcal{C}$-probe $P : C \to X$ and open subsets $U \ll V \subseteq C$, such that $S \subseteq P(U)$ and $P(V) \subseteq T$. Let Z be a $\mathcal{C}$-generated space, $W \subseteq Z \times_{\mathcal{C}} X$ be an open and assume that $\{z\} \times T \subseteq W$. Applying the map $id_Z \times P : Z \times_{\mathcal{C}} C \to Z \times_{\mathcal{C}} X$, we have $\{z\} \times V \subseteq \{z\} \times P^{-1}(T) \subseteq (id \times P)^{-1}(W)$. By Theorem 1.1 part (i), $Z \times_{\mathcal{C}} C = Z \times C$, and hence there is a neighborhood G of z, such that $G \times U \subseteq (id \times P)^{-1}(W)$. Thus $G \times S \subseteq W$, as desired.

(ii) Let Z be a $\mathcal{C}$-generated space, $W \subseteq Z \times_{\mathcal{C}} X$ be an open and assume that $\{z\} \times T \subseteq W$. By Theorem 1.1 part (i), $Z \times_{\mathcal{C}} X = Z \times X$, so there is a neighborhood V of z, such that $V \times S \subseteq W$, as desired.

Recall that a $\mathcal{C}$-generated space X is called $\mathcal{C}$-Hausdorff, if its diagonal is closed in $X \times_{\mathcal{C}} X$, and that is $\mathcal{C}$-discrete, if its diagonal is open in $X \times_{\mathcal{C}} X$ [5].

Theorem 2.1. Let X and Y be $\mathcal{C}$-generated spaces.

(i) If F is a closed set of X and $F \ll^{\mathcal{C}} X$, then F is $\mathcal{C}$-compact.

(ii) If X is $\mathcal{C}$-Hausdorff and $S \ll^{\mathcal{C}} T$ in X, then $\bar{S} \subseteq T$.

(iii) If $f : X \to Y$ is a map and $S \ll^{\mathcal{C}} T$ in X, then $f(S) \ll^{\mathcal{C}} f(T)$ in Y.

(iv) If $S \ll^{\mathcal{C}} T$ in X and $A \ll^{\mathcal{C}} B$ in Y, then $S \times A \ll^{\mathcal{C}} T \times B$ in $X \times_{\mathcal{C}} Y$.

Proof. The proofs are modifications of the classical proofs, so we prove only part (iv). Let Z be a $\mathcal{C}$-generated space, $W \subseteq Z \times_{\mathcal{C}} X \times_{\mathcal{C}} Y$ be an open and assume that $\{z\} \times T \times B \subseteq W$. Then for every $t \in T$, $\{(z,\ t)\} \times B \subseteq W$, so there is a neighborhood V_t of $(z,\ t)$, such that $V_t \times A \subseteq W$. Now let $W' = \bigcup_{t \in T} V_t$. Then $W' \subseteq Z \times_{\mathcal{C}} X$ is open and $\{z\} \times T \subseteq W'$. Thus there is a neighborhood V_z of z, such that $V_z \times S \subseteq W'$, and hence $V_z \times S \times A \subseteq W' \times A \subseteq W$, as desired.

The Sierpinski space is the space $\mathbb{S}$ with two points 1 and 0, such that $\{1\}$ is open but $\{0\}$ is not. We denote by $O_{\mathcal{C}}X$ the lattice of open sets of X endowed with the topology that makes the bijection $U \mapsto \chi_U : O_{\mathcal{C}}X \to \mathbb{S}^X$ into a homeomorphism, where the exponential is calculated in $Top_{\mathcal{C}}$. We henceforth assume that the Sierpinski space is a $\mathcal{C}$-generated space. The following lemma can be found in [5].

Lemma 2.1. Let X be a $\mathcal{C}$-generated space.

(i) The topology of $O_{\mathcal{C}}X$ is finer than the Scott topology.

(ii) The topology of $O_{\mathcal{C}}X$ coincides with the Scott topology if $\mathcal{C}$ generates all compact Hausdorff spaces.

(iii) A set $W \subseteq Z \times_{\mathcal{C}} X$ is open if and only if its transpose $\bar{W} : Z \to O_{\mathcal{C}}X$ defined by $\bar{W}(z) = \{x \in X \mid (y,\ x) \in W\}$ is continuous.

(iv) A set $W \subseteq Z \times_{\mathcal{C}} X$ is open if and only if for every $x \in X$, the set $V_x = \{z \in Z \mid (z,\ x) \in W\}$ is open and also for every Scott open set O of OZ, the set $U_O = \{x \in X \mid V_x \in O\}$ is open.

(v) The set $\{(U,\ x) \in O_{\mathcal{C}}X \times_{\mathcal{C}} X \mid x \in U\}$ is open in the $\mathcal{C}$-product.

For a $\mathcal{C}$-generated space X and an arbitrary subset S of X, we denote by O_S the set $\{U \in O_{\mathcal{C}}X \mid S \subseteq U\}$ of $O_{\mathcal{C}}X$.

Lemma 2.2. If W is an open set of $Z \times_{\mathcal{C}} X$, then for every subset S of X, $\bar{W}^{-1}(O_S) = \{z \in Z \mid \{z\} \times S \subseteq W\}$, where $\bar{W}$ is the map defined in Lemma 2.1.

Proof.

$$\begin{aligned} z \in \bar{W}^{-1}(O_S) &\Leftrightarrow \bar{W}(z) \in O_S \\ &\Leftrightarrow S \subseteq \bar{W}(z) \\ &\Leftrightarrow \{z\} \times S \subseteq W \\ &\Leftrightarrow z \in \{z \in Z \mid \{z\} \times S \subseteq W\}. \end{aligned}$$

Theorem 2.2. Let X be a $\mathcal{C}$-generated space. Then the following are equivalent:

(i) $S \ll^{\mathcal{C}} T$ in X.

(ii) $O_T \subseteq {O_S}^{\circ}$, where ${O_S}^{\circ}$ is the interior of O_S in the space $O_{\mathcal{C}}X$.

Proof. (i)⇒(ii) Let $U \in O_T$. By Lemma 2.1, the set $W = \{(U,\ x) \in O_{\mathcal{C}}X \times_{\mathcal{C}} X \mid x \in U\}$ is open and $\{U\} \times T \subseteq W$. Thus there is an open neighborhood O of U, such that $O \times S \subseteq W$, which shows that $U \in {O_S}^{\circ}$, as desired.

(ii)⇒(i) Let Z be a $\mathcal{C}$-generated space, $W \subseteq Z \times_{\mathcal{C}} X$ be an open. By assumption, $\bar{W}^{-1}(O_T) \subseteq \bar{W}^{-1}({O_S}^{\circ})$, where $\bar{W}$ is the map defined in Lemma 2.1. Thus by Lemma 2.2, $\{z \in Z \mid \{z\} \times T \subseteq W\} \subseteq \{z \in Z \mid \{z\} \times S \subseteq W\}^{\circ}$, and hence by Remark 2.1, the result follows.

Corollary 2.1. A subset Q of a $\mathcal{C}$-generated space X is $\mathcal{C}$-compact if and only if the set $\{U \in O_{\mathcal{C}}X \mid Q \subseteq U\}$ is open in the space $O_{\mathcal{C}}X$.

Proof. By Theorem 2.2, we have that

$$Q \ll^{\mathcal{C}} Q \Leftrightarrow O_Q \subseteq {O_Q}^{\circ} \Leftrightarrow O_Q = {O_Q}^{\circ}.$$

Corollary 2.2. Every compact set is $\mathcal{C}$-compact and if the class $\mathcal{C}$ generates all compact Hausdorff spaces, the converse holds.

Proof. By definition of the Scott topology a subset Q of a $\mathcal{C}$-generated space X is compact if and only if the set $\{U \in O_{\mathcal{C}}X \mid Q \subseteq U\}$ is Scott open. Thus by Lemma 2.1 and Corollary 2.1, the result follows.

§3. Proper maps

In this section we present some properties of proper maps in $Top_{\mathcal{C}}$, in particular, we show that f is a proper map in one of the categories, core compactly generated spaces, locally compactly generated spaces, compactly generated spaces, if and only if it is a proper map in the category Top of topological spaces.

Proposition 3.1. If $\mathcal{D}$ and $\mathcal{C}$ are two productive classes of generating spaces such that $\mathcal{D} \subseteq \mathcal{C}$, then the inclusion functor $i : Top_{\mathcal{D}} \to Top_{\mathcal{C}}$ reflects proper maps.

Proof. By Theorem 1.1 part (ii), the finite products in the categories $Top_{\mathcal{C}}$ and $Top_{\mathcal{D}}$ coincide. Thus if $f \in Top_{\mathcal{D}}$ is a proper map in $Top_{\mathcal{C}}$, then it is a proper map in $Top_{\mathcal{D}}$.

Remark 3.1. A map $g : A \to B$ is closed if and only if for every open set $U \subseteq A$, the set $B \setminus g(A \setminus U)$ is open. But an easy calculation shows that this set is $\{b \in B \mid g^{-1}\{b\} \subseteq U\}$. Thus a map $g : A \to B$ is closed if and only if for every open set $U \subseteq A$, the set $\{b \in B \mid g^{-1}\{b\} \subseteq U\}$ is open.

Theorem 3.1. A map $f : X \to Y$ is proper in $Top_{\mathcal{C}}$ if and only if for every $\mathcal{C}$-generated space Z and every open set $W \subseteq Z \times_{\mathcal{C}} X$, the set $\{(z,\ y) \in Z \times_{\mathcal{C}} Y \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$ is open.

Proof. By Remark 3.1, we have that f is a proper map if and only if for every $\mathcal{C}$-generated space Z and every open set $W \subseteq Z \times_{\mathcal{C}} X$, the set $\{(z,\ y) \in Z \times_{\mathcal{C}} Y \mid (id_Z \times f)^{-1}\{(z,\ y)\} \subseteq W\}$ is open.

Definition 3.1. Let $f : X \to Y$ be a map in $Top_{\mathcal{C}}$. We say that f reflects relative $\mathcal{C}$-compactness, if for arbitrary subsets S and T of Y, such that $S \ll^{\mathcal{C}} T$, then $f^{-1}(S) \ll^{\mathcal{C}} f^{-1}(T)$.

Theorem 3.2. If $f : X \to Y$ is a proper map in $Top_{\mathcal{C}}$, then f reflects relative $\mathcal{C}$-compactness.

Proof. Let $S \ll^{\mathcal{C}} T$ in Y, W be an open subset of $Z \times_{\mathcal{C}} X$, and $z \in Z$ such that, $\{z\} \times f^{-1}(T) \subseteq W$. Then by Theorem 3.1, the set $M = \{(z,\ y) \mid \{z\} \times f^{-1}(y) \subseteq W\}$ is open. Since $\{z\} \times T \subseteq M$, by definition of relative $\mathcal{C}$-compactness, $V \times S \subseteq M$ for some neighborhood V of z. Therefore $V \times f^{-1}(S) \subseteq W$, as desired.

Corollary 3.1. If $f : X \to Y$ is a proper map in $Top_{\mathcal{C}}$, then:

(i) f is a closed map and the set $f^{-1}(Q)$ is $\mathcal{C}$-compact, for every $\mathcal{C}$-compact set Q of Y.

(ii) f is a closed map and the set $f^{-1}\{y\}$ is $\mathcal{C}$-compact, for every point y in Y.

Proof. Since the one-point space is a $\mathcal{C}$-generated space, so if Z is the one-point space, then every proper map is closed. Singletons are compact and hence are $\mathcal{C}$-compact, so by Theorem 3.2, the result follows.

We know that the converse of Corollary 3.1 in Top holds, but we show that if the class $\mathcal{C}$ is productive, then the converse of Corollary 3.1 in $Top_{\mathcal{C}}$ holds.

Remark 3.2. Recall that the saturation of a productive $\mathcal{C}$ is the collection $\bar{\mathcal{C}}$ of $\mathcal{C}$-generated spaces which are exponentiable in Top. A space is $\mathcal{C}$-generated if and only if it is a quotient of a space in $\bar{\mathcal{C}}$, see Lemma 5.3 in [7].

Lemma 3.1. Let $f : X \to Y$ be a closed map in $Top_{\mathcal{C}}$, such that the set $f^{-1}\{y\}$ is $\mathcal{C}$-compact, for every point y in Y. Then for every $\mathcal{C}$-generated space Z, which is exponentiable in Top and every open set $W \subseteq Z \times_{\mathcal{C}} X$, the set $\{(z,\ y) \in Z \times_{\mathcal{C}} Y \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$ is open.

Proof. Let $Z \in Top_{\mathcal{C}}$ be an exponentiable space and $W \subseteq Z \times_{\mathcal{C}} X$ be open and assume that $M = \{(z,\ y) \in Z \times_{\mathcal{C}} Y \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$. By Remark 2.1, the set $V_y = \{z \in Z \mid (z,\ y) \in M\} = \{z \in Z \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$ is open, for every y in Y. To show that M is open, let O be a Scott open set of OZ. By Lemma 2.1 part (iv), it suffices to show that the set $U_O = \{y \in Y \mid V_y \in O\}$ is open. Let $y_0 \in Y$ be such that $V_{y_o} \in O$. Since Z is exponentiable,

for every $z \in V_{y_o}$, there is an open neighborhood U_z of z such that $U_z \ll V_{y_o}$. By definition of the Scott topology, there is a finite set I of V_{y_o}, such that $\bigcup_{z\in I} U_z \in O$. Suppose that $O_z = \{V \in OZ \mid U_z \ll V\}$. By Remark 3.1, the set $G = \{y \in Y \mid f^{-1}\{y\} \subseteq \bigcap_{z\in I} \bar{W}^{-1}(O_z)\}$ is an open neighborhood of y_0, where $\bar{W}$ is the map defined in Lemma 2.1. Now we show that $G \subseteq U_O$. Let $y \in G$. Then for every $x \in f^{-1}\{y\}$ and every $z \in I$, $U_z \ll \bar{W}(x)$, and hence $\bigcup_{z\in I} U_z \subseteq \bigcap_{x\in f^{-1}\{y\}} \bar{W}(x) = V_y$. Therefore $V_y \in O$, as desired.

Theorem 3.3. Let $f : X \to Y$ be a map in $Top_{\mathcal{C}}$. If the class $\mathcal{C}$ is productive, then the following are equivalent:

(i) f is a proper map in $Top_{\mathcal{C}}$.

(ii) For every $\mathcal{C}$-generated space Z and every open set $W \subseteq Z \times_{\mathcal{C}} X$, the set $\{(z,\ y) \in Z \times_{\mathcal{C}} Y \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$ is open.

(iii) f is a closed map and the set $f^{-1}(Q)$ is $\mathcal{C}$-compact, for every $\mathcal{C}$-compact set Q of Y.

(iv) f is a closed map and the set $f^{-1}\{y\}$ is $\mathcal{C}$-compact, for every point y in Y.

Proof. By Theorem 3.1 and Corollary 3.1, (i)⇔(ii)⇒(iii)⇒(iv).

(iv)⇒(ii) Let $Z \in Top_{\mathcal{C}}$ and $W \subseteq Z \times_{\mathcal{C}} X$ be an open subset and assume that $M = \{(z,\ y) \in Z \times_{\mathcal{C}} Y \mid \{z\} \times f^{-1}\{y\} \subseteq W\}$. By Remark 3.2, there is a quotient map $q : C \to Z$, with $C \in \bar{\mathcal{C}}$. By Lemma 3.1, the set $G = \{(c,\ y) \in C \times_{\mathcal{C}} Y \mid \{c\} \times f^{-1}\{y\} \subseteq (q \times id_X)^{-1}(W)\}$ is open. But an easy calculation shows that $G = (q \times id_Y)^{-1}(M)$, by Theorem 1.1 part (iii), $q \times id_Y$ is a quotient map, and hence the set M is open, as desired.

Corollary 3.2. (i) If the class $\mathcal{C}$ generates all compact Hausdorff spaces, then the inclusion functor $i : Top_{\mathcal{C}} \to Top$ preserves proper maps.

(ii) If the class $\mathcal{C}$ is productive, then the inclusion functor $i : Top_{\mathcal{C}} \to Top$ reflects proper maps.

Proof. (i) Let $f \in Top_{\mathcal{C}}$ be a proper map. Then by Corollary 3.1, f is closed and the set $f^{-1}\{y\}$ is $\mathcal{C}$-compact, for every point $y \in Y$. By Corollary 2.2, every $\mathcal{C}$-compact set is compact, so f is a proper map in Top.

(ii) By Corollary 2.2 and Theorem 3.3, the result follows.

Corollary 3.3. Since every compact Hausdorff space is locally compact and every locally compact space is core compact, therefore by Corollary 3.2, f is a proper map in one of the categories, core compactly generated spaces, locally compactly generated spaces, compactly generated spaces, if and only if it is a proper map in the category Top of all topological spaces.

References

[1] N. Bourbaki, General topology, Addison-Wesley, London, **1**(1988).

[2] M. M. Clementino, E. Giuli and W. Tholen, Topology in a category: compacness, Portugaliae Mathematica, **53**(1996), 397-433.

[3] M. M. Clementino and D. Hofmann, On some special classes of continouous maps, Open problems in topology II, Chaper 4, (2007), 367-376.

[4] M. M. Clementino, D. Hofmann and W. Tholen, The convergence approach to exponentiable maps, Portugaliae Mathematica, **60**(2003), 139-160.

[5] M. H. Escardó, Notes on compactness, University of Birmingham, unpublished notes, 8th April 2005.

[6] M. H. Escardó and R. Heckmann, Topologies on spaces of continuous functions, Topology Proceedings, **26**(2001-2002), No. 2, 545-565.

[7] M. H. Escardó, J. D. Lawson and A. Simpson, Comparing cartesian closed categories of (core) compactly generated spaces, Topology and its Applications, **143**(2004), 105-145.

[8] G. Richter and W. Tholen, Perfect maps are exponentiable-categorically, Theory and Applications of Categories, **16**(2001), No. 8, 457-464.

[9] W. Tholen, A categorical guide to separation, compactness and perfectness, Homology, Homotopy and Applicatins, **1**(1999), 147-161.

ISBN 9781599731810

International Book Series